THE NILPOTENT FILTRATION AND THE ACTION OF AUTOMORPHISMS ON THE COHOMOLOGY OF FINITE p–GROUPS

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ABSTRACT. We study $H^*(P)$, the mod p cohomology of a finite p–group P, viewed as an $\mathbb{F}_p[Out(P)]$–module. In particular, we study the conjecture, first considered by Martino and Priddy, that, if $e_S \in \mathbb{F}_p[Out(P)]$ is a primitive idempotent associated to an irreducible $\mathbb{F}_p[Out(P)]$–module S, then the Krull dimension of $e_S H^*(P)$ equals the rank of P. The rank is an upper bound by Quillen's work, and the conjecture can be viewed as the statement that every irreducible $\mathbb{F}_p[Out(P)]$–module occurs as a composition factor in $H^*(P)$ with similar frequency.

In summary, our results are as follows. A strong form of the conjecture is true when p is odd. The situation is much more complex when $p = 2$, but is reduced to a question about 2–central groups (groups in which all elements of order 2 are central), making it easy to verify the conjecture for many finite 2–groups, including all groups of order 128, and all groups that can be written as the product of groups of order 64 or less.

The the odd prime theorem can be deduced using the approach to $\mathcal{U}$, the category of unstable modules over the Steenrod algebra, initiated by H.-W. Henn, J. Lannes, and L. Schwartz in [HLS1]. The reductions when $p = 2$ make heavy use of the nilpotent filtration of $\mathcal{U}$ introduced in [S1], as applied to group cohomology in [HLS2]. Also featured are unstable algebras of cohomology primitives associated to central group extensions.

1. INTRODUCTION

Fix a prime p, and let $H^*(P)$ denote the mod p cohomology ring of a finite p–group P. Interpreted topologically, $H^*(P) = H^*(BP; \mathbb{F}_p)$, where BP is the classifying space of P, and interpreted algebraically, $H^*(P) = \text{Ext}^*_{\mathbb{F}_p[P]}(\mathbb{F}_p, \mathbb{F}_p)$.

The automorphism group of P, $Aut(P)$, acts on P, and thus via ring homomorphisms on $H^*(P)$. As the inner automorphism group, $Inn(P)$, acts trivially on cohomology, $H^*(P)$ becomes a graded $\mathbb{F}_p[Out(P)]$–module, where $Out(P) = Aut(P)/Inn(P)$ is the outer automorphism group.

The ring $H^*(P)$ is known to be Noetherian, and D. Quillen [Q1] computed its Krull dimension: $\dim H^*(P) = \text{rk}(P)$. Here $\text{rk}(P)$ denotes the maximal rank of an elementary abelian p–subgroup of P.

Now let S be an irreducible $\mathbb{F}_p[Out(P)]$–module and $e_S \in \mathbb{F}_p[Out(P)]$ an associated primitive idempotent. Then $e_S H^*(P)$ is a finitely generated module over the Noetherian ring $H^*(P)^{Out(P)}$ (see §4.3), and thus has a Krull dimension with evident upper bound $\text{rk}(P)$. In the early 1990's, J. Martino and S. Priddy [MP]

asked whether the following conjecture might be true.

Conjecture A $\dim e_S H^*(P) = \mathrm{rk}(P)$ for all pairs (P, S).

As $\dim e_S H^*(P)$ determines the growth of the Poincaré series of $e_S H^*(P)$, and this Poincaré series gives a count of the occurrences of S as a composition factor in $H^*(P)$, the conjecture is roughly the statement that every irreducible $\mathbb{F}_p[Out(P)]$–module occurs in $H^*(P)$ with similar frequency.

The topological interpretation of $H^*(P)$ makes it evident that $H^*(P)$ is an object in $\mathcal{K}$ and $\mathcal{U}$, the categories of unstable algebras and modules over the mod p Steenrod algebra $\mathcal{A}$, and in this paper, we study Conjecture A using modern '$\mathcal{U}$–technology'.

In summary, our results are as follows. A strong form of Conjecture A is true when p is odd. The situation is more complex when $p = 2$. We reduce the conjecture to a different conjecture about 2–central groups, making it easy to verify Conjecture A for many finite 2–groups, including all with order dividing 128, and all groups that can be written as the product of groups of order 64 or less.

The the odd prime theorem can be deduced using the functor category approach to $\mathcal{U}$ initiated by H.-W. Henn, J. Lannes, and L. Schwartz in [HLS1]. The reductions when $p = 2$ make heavy use of the nilpotent filtration of $\mathcal{U}$ introduced in [S1], as applied to group cohomology in [HLS2].

Featured here, and in a companion paper [K5], are algebras of cohomology primitives associated to central group extensions, and p–central groups: groups in which every element of order p is central.

2. Main results

2.1. **Notation.** Before describing our results in more detail, we introduce some notation. Throughout, $V < P$ will denote an elementary abelian p–subgroup of P and $C = C(P) < P$ will be the maximal central elementary abelian p–subgroup. We let $c(P)$ be the rank of C, $\mathrm{depth}_{H^*(P)} H^*(P)$ be the depth of $H^*(P)$ with respect to the ideal of positive degree elements, and $\mathrm{mrk}(P)$ be the minimal rank of a maximal $V < P$. These are related by

$$c(P) \leq \mathrm{depth}_{H^*(P)} H^*(P) \leq \mathrm{mrk}(P) \leq \mathrm{rk}(P),$$

where the first inequality is due to J.Duflot [D], and the second follows from Quillen's work and standard commutative algebra (see §4.6).

Given $V < P$, we let $Aut_0(P, V) \triangleleft Aut(P, V) < Aut(P)$ be defined by

$$Aut(P, V) = \{\alpha \in Aut(P) \mid \alpha(V) = V\} \text{ and}$$

$$Aut_0(P, V) = \{\alpha \in Aut(P) \mid \alpha(v) = v \text{ for all } v \in V\},$$

Then let $Out_0(P, V)$ and $Out(P, V)$ be the projection of these groups in $Out(P)$. Note that $Out(P, V) = Aut(P, V)/N_P(V)$ and $Out_0(P, V) = Aut_0(P, V)/C_P(V)$, so that there is an extension of groups

$$W_P(V) \to Aut(P, V)/Aut_0(P, V) \to Out(P, V)/Out_0(P, V),$$

where $W_P(V) = N_P(V)/C_P(V)$.

Finally, if S is an irreducible $\mathbb{F}_p[G]$–module, and M is an arbitrary $\mathbb{F}_p[G]$–module, we say that S *occurs in* M if it is a composition factor.

2.2. **Quillen's approximation.** Let $\mathcal{A}(P)$ denote Quillen's category, having as objects the elementary abelian p–subgroups $V < P$, and as morphisms the homomorphisms generated by inclusions and conjugation by elements in P. Let $\mathcal{H}^*(P)$ denote $\mathcal{H}^*(P) = \lim_{\mathcal{A}(P)} H^*(V)$. It is not hard to see the same inverse limit is attained by using the smaller category $\mathcal{A}^C(P)$, where $\mathcal{A}^C(P) \subset \mathcal{A}(P)$ is the full subcategory consisting of elementary abelian $V < P$ containing the maximal central elementary abelian C.

The restriction maps associated to the inclusions $V < P$ assemble to define a natural map of unstable $\mathcal{A}$–algebras

$$H^*(P) \to \mathcal{H}^*(P),$$

which Quillen shows is an F–isomorphism. As a special case of Proposition 2.6 below, we have

$$\dim e_S H^*(P) \geq \dim e_S \mathcal{H}^*(P),$$

for all irreducible $\mathbb{F}_p[Out(P)]$–modules S.

Theorem 2.1. *Let S be an irreducible $\mathbb{F}_p[Out(P)]$–module. Then $e_S \mathcal{H}^*(P) \neq 0$ if and only if S occurs in $\mathbb{F}_p[Out(P)/Out_0(P,V)]$ for some maximal $V < P$. In that case, $\dim e_S \mathcal{H}^*(P) = \max\{\mathrm{rk}(V) \mid S$ occurs in $\mathbb{F}_p[Out(P)/Out_0(P,V)]\}$.*

This implies an observation noted earlier by Martino and Priddy [MP, Prop.4.2].

Corollary 2.2. *If $e_S \mathcal{H}^*(P) \neq 0$, then $\dim e_S \mathcal{H}^*(P) \geq \mathrm{mrk}(P)$.*

Given $H < G$, a standard argument shows that if H is a p–group, then every irreducible $\mathbb{F}_p[G]$–module occurs as a submodule of $\mathbb{F}_p[G/H]$. Thus the theorem implies that if $Out_0(P,V)$ is a p–group, then $\dim e_S \mathcal{H}^*(P) \geq \mathrm{rk}(V)$ for all irreducibles S. The converse will be true if P has a unique maximal elementary abelian subgroup, for example, if P is p–central: see Corollary 6.3.

One can now make effective use of group theory results about p'–automorphisms of p–groups, particularly the Thompson $A \times B$ lemma [Gor, Thm.5.3.4].

Theorem 2.3. *Let P a finite p–group. If $V < P$ is an elementary abelian subgroup, then $Out_0(P,V)$ will be a p–group if $Out_0(C_P(V),V)$ is a p–group. If V is also maximal, and p is odd, this will always be the case. Thus if p is odd, $\dim e_S \mathcal{H}^*(P) = \mathrm{rk}(P)$ for all irreducible $\mathbb{F}_p[Out(P)]$–modules S.*

For 2–groups, one does not need to look far to see that story is quite different. The quaternionic group of order 8, Q_8, is 2–central with center C of rank 1. As $Aut(Q_8)$ clearly must act trivially on the center, $Out_0(Q_8, C) = Out(Q_8) \simeq \Sigma_3$, which is not a 2–group.

This example generalizes: it is the case $t = 1$ in the following.

Example 2.4. Let G_t be the 2–Sylow subgroup of $SU_3(\mathbb{F}_{2^t})$. This group is 2–central with center $C_t \simeq \mathbb{F}_{2^t}$, and $G_t/C_t \simeq \mathbb{F}_{2^{2t}}$. The group $Out(G_t, C_t)$ contains a cyclic subgroup of order $2^t + 1$. Thus there is at least one irreducible $\mathbb{F}_2[Out(G_t)]$–module S with $e_S \mathcal{H}^*(G_t) = 0$.

The next proposition, an easy application of properties of Lannes T_V–functor, shows that Theorem 2.3 can be applied to 2–groups with reduced cohomology, i.e., groups P for which $H^*(P) \to \mathcal{H}^*(P)$ is monic. This includes groups built up by iterated products and wreath products of $\mathbb{Z}/2$'s.

Proposition 2.5. *If $H^*(P)$ is reduced, and $V < P$ is a maximal elementary abelian subgroup, then $C_P(V) = V$, and thus $Out_0(C_P(V), V)$ is the trivial group.*

To say more about $\dim e_S H^*(P)$, we now begin to take a deeper look at $H^*(P)$ using the nilpotent filtration.

2.3. **A stratification of the problem.** In §4, we study functors $\bar{R}_d : \mathcal{U} \to \mathcal{U}$. In brief, they are defined as follows. An unstable module M has a natural nilpotent filtration

$$\cdots \subset nil_2 M \subset nil_1 M \subset nil_0 M = M,$$

and $nil_d M / nil_{d+1} M = \Sigma^d R_d M$, where $R_d M$ is a reduced unstable module. Then $\bar{R}_d M$ is defined as the nilclosure of $R_d M$.

Henn [H] has shown that the nilpotent filtration of a Noetherian unstable algebra is finite, so only finitely many of the the modules $\bar{R}_d H^*(P)$ will be nonzero. Furthemore, $\bar{R}_0 H^*(P) = \mathcal{H}^*(P)$, and each $\bar{R}_d H^*(P)$ is a finitely generated $\mathcal{H}^*(P)$–module. The study of $\dim e_S H^*(P)$ stratifies as follows.

Proposition 2.6. *Let S be an irreducible $\mathbb{F}[Out(P)]$–module. Then*

$$\dim e_S H^*(P) = \max_d \{\dim e_S \bar{R}_d H^*(P)\}.$$

This will be a special case of a more general statement about $\mathcal{K}$: see Proposition 4.9.

Martino and Priddy observe [MP, Prop.4.1] that the Depth Conjecture, subsequently proved by D. Bourguiba and S. Zarati [BoZ], implies that $\dim e_S H^*(P) \geq \mathrm{depth}_{H^*(P)} H^*(P)$. We prove a stratified variant of this, and note that an analogue of Duflot's theorem holds.

Proposition 2.7. *If $e_S \bar{R}_d H^*(P) \neq 0$, then*

$$\dim e_S \bar{R}_d H^*(P) \geq \mathrm{depth}_{\mathcal{H}^*(P)} \bar{R}_d H^*(P) \geq c(P).$$

That $\dim e_S \bar{R}_d H^*(P) \geq c(P)$ is also a corollary of part (c) of Theorem 2.8 below.

2.4. **Computing** $\dim e_S \bar{R}_d H^*(P)$. In [K5], the work of [HLS2] will allow us to write down useful formulae for $\bar{R}_d H^*(P)$ analogous to

$$\bar{R}_0 H^*(P) = \lim_{V \in \mathcal{A}^C(P)} H^*(V).$$

Using these, we can generalize Theorem 2.1 to a statement about $\dim e_S \bar{R}_d H^*(P)$ for all d.

To state this, we need to define the primitives associated to a central extension.

The cohomology of an elementary abelian p–group is a Hopf algebra. If Q is a finite group, and $V < Q$ is a central elementary abelian p–subgroup, then multiplication $m : V \times Q \to Q$ is a homomorphism, and the induced map in cohomology, $m^* : H^*(Q) \to H^*(V \times Q) = H^*(V) \otimes H^*(Q)$ makes $H^*(Q)$ into an $H^*(V)$–comodule. We then let $P_V H^*(Q)$ denote the primitives:

$$P_V H^*(Q) = \{x \in H^*(Q) \mid m^*(x) = 1 \otimes x\}$$

$$= \mathrm{Eq} \, \{H^*(Q) \xrightarrow[\pi^*]{m^*} H^*(V \times Q)\},$$

where $\pi : V \times Q \to Q$ is the projection.

As equalizers of algebra maps are algebras, $P_V H^*(Q)$ is again an unstable $\mathcal{A}$–algebra. Note also that $P_V H^0(Q) = \mathbb{F}_p$. It is not hard to check that $P_V H^1(Q) \simeq H^1(Q/V) = \mathrm{Hom}(Q/V, \mathbb{F}_p)$, via the inflation map. The reader may find it illuminating to know that $P_V H^*(Q)$ is again Noetherian, and has Krull dimension equal to $\mathrm{rk}(Q) - \mathrm{rk}(V)$: see [K4, K5].

Our theorem about $\dim e_S \bar{R}_d H^*(P)$ now goes as follows.

Theorem 2.8. *Let S be an irreducible $\mathbb{F}_p[Out(P)]$–module. For each d, there exists a set $\sup_d(S) \subset ob\ \mathcal{A}^C(P)$, the d–support of S, with the following properties.*

(a) $\sup_d(S)$ *is a union of $Aut(P)$–orbits in $ob\ \mathcal{A}^C(P)$.*

(b) *Suppose that $V_1 < V_2 < P$, and $P_{V_1} H^d(C_P(V_1)) \to P_{V_1} H^d(C_P(V_2))$ is monic. Then $V_1 \in \sup_d(S)$ implies that $V_2 \in \sup_d(S)$.*

(c) $e_S \bar{R}_d H^*(P) \neq 0$ *if and only if $\sup_d(S) \neq \emptyset$, and, in this case,*

$$\dim e_S \bar{R}_d H^*(P) = \max\{\mathrm{rk}(V) \mid V \in \sup_d(S)\}.$$

(d) *Let $V \in ob\ \mathcal{A}^C(P)$ be maximal. Then $V \in \sup_d(S)$ if and only if S occurs in $\mathrm{Ind}_{Out(P,V)}^{Out(P)}([\mathbb{F}_p[Aut(P,V)/Aut_0(P,V)] \otimes P_V H^d(C_P(V))]^{W_P(V)}).$*

When $d = 0$, we recover Theorem 2.1. In this case, the hypothesis in part (b) always holds. Thus maximal elements in $sup_0(S)$ will always be maximal in $ob\ \mathcal{A}^C(P)$. Thus part (d) implies that V will be a maximal element in $\sup_0(S)$ if and only if S occurs in

$$\mathrm{Ind}_{Out(P,V)}^{Out(P)}(\mathbb{F}_p[Aut(P,V)/Aut_0(P,V)]^{W_P(V)})$$

which rewrites as

$$\mathbb{F}_p[Out(P,V)/Out_0(P,V)].$$

Thus part (c) implies the calculation of $\dim e_S \mathcal{H}^*(P)$ in Theorem 2.1.

Theorem 2.8 now lets us recast Conjecture A as follows.

Corollary 2.9. *Conjecture A is true for a pair (P, S) if and only if there exists a $V < P$ of maximal rank such that S occurs in*

$$\mathrm{Ind}_{Out(P,V)}^{Out(P)}([\mathbb{F}_p[Aut(P,V)/Aut_0(P,V)] \otimes P_V H^*(C_P(V))]^{W_P(V)}).$$

Starting from this, it is not hard to show (see Lemma 8.2) that Conjecture A for a 2–group P would be implied by

Conjecture B Let P be a finite 2–group. If $V < P$ is a maximal elementary abelian subgroup, then every irreducible $\mathbb{F}_2[Out_0(P,V)]$–module occurs in $P_V H^*(C_P(V))^{W_P(V)}$.

Note that the subgroups $C_P(V)$ that arise here are 2–central. Our general theory shows

Proposition 2.10. *If Q is 2–central with maximal elementary abelian subgroup C, then Conjecture B is true for Q, i.e. every irreducible $\mathbb{F}_2[Out_0(Q,C)]$–module occurs as a composition factor in $P_C H^*(Q)$.*

We conjecture a subtle strengthening of this.

Conjecture C If Q is 2–central with maximal elementary abelian subgroup C, then every irreducible $\mathbb{F}_2[Out_0(Q,C)]$–module occurs as a submodule of $P_C H^*(Q)$.

Again using the Thompson $A \times B$ lemma, we have

Theorem 2.11. *Conjecture B is true for the pair (P,V) if the 2–central group $C_P(V)$ satisfies Conjecture C. Thus if P is a 2–group such that $C_P(V)$ satisfies Conjecture C for some $V < P$ of maximal rank, then $\dim e_S H^*(P) = \mathrm{rk}(P)$ for all irreducible $\mathbb{F}_2[Out(P)]$–modules S.*

Though $Out_0(Q,C)$ need not be a 2–group if Q is 2–central, its $2'$ part tends to be very small, even in cases when $Out(Q)$ is quite complicated. This makes it not hard to check the following theorem, using information about 2–groups of order dividing 64 available at the website [Ca], or, in book form, [CTVZ].

Theorem 2.12. *Conjecture C is true for all 2–central groups that can be written as the product of groups of order dividing 64.*

A counterexample to Conjecture A would have to be a 2–group P that is not 2–central, and having the property that the proper subgroups $C_P(V)$, with V of maximal rank, are all counterexamples to Conjecture C. Thus the last theorem implies

Corollary 2.13. *Conjecture A is true for all 2–groups of order dividing 128, and all 2–groups that can be written as the product of groups of order 64 or less.*

We end this section with an example[1] that illustrates the sorts of patterns that the numbers $\dim e_S \bar{R}_d H^*(P)$ can take, when P is not p–central.

Example 2.14. Let P be the group of order 64 number #108 on the Carlson group cohomology website. From the information there one learns that $Out(P)$ has order $3 \cdot 2^8$, $c(P) = \mathrm{depth}_{H^*(P)} H^*(P) = 2$, while $\mathrm{mrk}(P) = \mathrm{rk}(P) = 3$. One can also deduce that $\mathbb{F}_2[Out(P)]$ has precisely two irreducibles - the one dimensional trivial module '1' and a two dimensional module 'S'. We compute the following table of nonzero dimensions, where $\emptyset$ denotes that the corresponding summand of $\bar{R}_d(H^*(P))$ is 0.

d	$\dim e_1 \bar{R}_d H^*(P)$	$\dim e_S \bar{R}_d H^*(P)$
0	3	$\emptyset$
1	2	3
2	2	3
3	3	2
4	2	2
5	2	2
6	2	2
7	2	$\emptyset$

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[1]Checking Conjecture A for this group led us to the formulation and proof of Theorem 2.8.

Note that this example shows that, when $d > 0$, $e_S \bar{R}_d H^*(P) \neq 0$ does not imply that $\dim e_S \bar{R}_d H^*(P) \geq \mathrm{mrk}(P)$, in contrast to Corollary 2.2. Indeed, for this group, $\dim \bar{R}_d H^*(P) = 2 < 3 = \mathrm{mrk}(P)$ for $d = 4, 5, 6, 7$.

2.5. **Organization of the paper.** The rest of the paper is organized as follows. In §3, we quickly survey previous results related to Conjecture A. The nilpotent filtration is reviewed in §4, which has various relevant general results about $\mathcal{U}$ and $\mathcal{K}$ concerning both dimension and depth. In §5, we give a first proof of Theorem 2.1 using the functor category description of $\mathcal{U}/Nil$ introduced in [HLS1]. In §6, we then collect various results about p'–automorphisms of p–groups, and prove Theorem 2.3 and Proposition 2.5. In this section we also discuss the family of 2-central groups given in Example 2.4. In §7, we use a convenient formula for $\bar{R}_d H^*(P)$ from [K5] to prove Theorem 2.8. A short proof of Proposition 2.10 also appears there. That Conjecture C implies Conjecture A is shown in the short §8, and in §9 various 2–central groups are shown to satisfy Conjecture C, including those listed in Theorem 2.12. Finally, in §10, we will discuss Example 2.14 in detail.

3. Previous results

In the 1984 paper [DS], T. Diethelm and U. Stammbach gave a group theoretic proof that, if P is a finite p–group, every irreducible $\mathbb{F}_p[Out(P)]$–module occurs as a composition factor in $H^*(P)$. Independently and concurrently, it was noted explicitly in [HK], and implicitly in [N], that this same result was a consequence of the proof of the Segal Conjecture in stable homotopy theory. From both [DS] and [HK], one can conclude that every irreducible occurs an infinite number of times, so that $\dim e_S H^*(P) \geq 1$ for all irreducibles S.

The 1999 paper of P. Symonds [Sy] gives a second group theoretic proof that every irreducible $\mathbb{F}_p[Out(P)]$–module occurs in $H^*(P)$, and inspection of his proof also yields Martino and Priddy's lower bound $\dim e_S H^*(P) \geq c(P)$.[2]

A proof of Conjecture A for an elementary abelian p–group V serves as a starting point for work on the general question, and goes as follows.

Let $S^*(V) \subset H^{2*}(V)$ denote the symmetric algebra generated by $\beta(H^1(V)) \subset H^2(V)$, where β is the Bockstein. Thus $S^*(V)$ is a polynomial algebra on $r = \mathrm{rk}(V)$ generators in degree 2. It is a classic result of Dickson [D, Cr] that the invariant ring $S^*(V)^{GL(V)}$ is again polynomial on homogeneous generators $c_V(1), \ldots, c_V(r)$, where $c_V(i)$ has degree $2(p^r - p^{r-i-1})$.

For any idempotent $e \in \mathbb{F}_p[GL(V)]$, $eH^*(V)$ is an $S^*(V)^{GL(V)}$–module via left multiplication. Now one observes that $S^*(V)$ (and thus $H^*(V)$) is a free

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[2][Sy] was written with knowledge of [HK], but apparently not of [DS].

$S^*(V)^{GL(V)}$–module, and that every finite $\mathbb{F}_p[GL(V)]$–module occurs as a sub-module of $S^*(V)$ [Alp, p.45]. These facts imply that, for all irreducible $\mathbb{F}_p[GL(V)]$–modules S, $e_S H^*(V)$ is a nonzero finitely generated free $S^*(V)^{GL(V)}$–module, and thus $\dim e_S H^*(V) = \mathrm{rk}(V)$.

Using [HK], this result immediately extends to all abelian p–groups.

In [MP], Martino and Priddy use the analogous action of $H^*(P)^{Out(P)}$ on $eH^*(P)$ to show that, for all nonzero idempotents e, $\dim eH^*(P) \geq \mathrm{depth}_{H^*(P)} H^*(P)$. Their proof is critically dependent on an unpublished paper that was later with-drawn. However, as we note in the next section, their argument can be salvaged by using the later work of D. Bourguiba and S. Zarati [BoZ] utilizing deep properties of $\mathcal{U}$ and $\mathcal{K}$.

Remark 3.1. Readers of [MP] will know that much of Martino and Priddy's paper concerns a refined version of Conjecture A. The double Burnside ring $A(P, P)$ acts on $H^*(P)$, and one can conjecture that $\dim eH^*(P) = \mathrm{rk}(P)$ for all idempotents $e \in A(P, P) \otimes \mathbb{F}_p$ that project to a nonzero element under the retraction of algebras $A(P, P) \otimes \mathbb{F}_p \to \mathbb{F}_p[Out(P)]$. The paper [HK] notes that one can deduce that $eH^*(P) \neq 0$, and thus that $\dim eH^*(P) \geq 1$, from the Segal conjecture; there is currently no 'group theoretic' proof of this fact. Partly for this reason, we have focused on the $Out(P)$ conjecture, though some of our general theory evidently applies to the $A(P, P)$ version.[3]

4. Krull dimension and the nilpotent filtration of $\mathcal{U}$

4.1. **The nilclosure functor.** Let $\mathcal{N}il_1 \subset \mathcal{U}$ be the localizing subcategory gener-ated by suspensions of unstable $\mathcal{A}$–modules, i.e. $\mathcal{N}il_1$ is the smallest full subcate-gory containing all suspensions of unstable modules that is closed under extensions and filtered colimits.

An unstable module M is called *nilreduced* (or just *reduced*) if it contains no such nilpotent submodules, or, equivalently, if $\mathrm{Hom}_{\mathcal{U}}(N, M) = 0$ for all $N \in \mathcal{N}il_1$. M is called *nilclosed* if also $\mathrm{Ext}^1_{\mathcal{U}}(N, M) = 0$ for all $N \in \mathcal{N}il_1$.

Let $L_0 : \mathcal{U} \to \mathcal{U}$ be localization away from $\mathcal{N}il_1$. Thus $L_0 M$ is nilclosed, and there is a natural transformation $M \to L_0 M$ with both kernel and cokernel in $\mathcal{N}il_1$.

Recall that $T_V : \mathcal{U} \to \mathcal{U}$ is defined to be left adjoint to $H^*(V) \otimes \underline{\hspace{1em}}$. The various marvelous properties of T_V are reflected in similar properties of L_0.

Proposition 4.1. *The functor $L_0 : \mathcal{U} \to \mathcal{U}$ satisfies the following properties.*

(a) *There are natural isomorphisms $L_0(M \otimes N) \simeq L_0 M \otimes L_0 N$.*

(b) *There are natural isomorphisms $T_V L_0 M \simeq L_0 T_V M$.*

(c) *If $K \in \mathcal{K}$, then $L_0 K \in \mathcal{K}$, and $K \to L_0 K$ is a map of unstable algebras.*

One approach to properties (a) and (b) is to use Proposition 5.1. See [HLS2, I.4.2] and [BrZ1] for more detail about property (c).

Given a Noetherian unstable algebra $K \in \mathcal{K}$, we recall the category $K_{f.g.} - \mathcal{U}$ as studied in [HLS2, I.4]. The objects are finitely generated K–modules M whose

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[3]Our results do imply the refined conjecture for p–central groups at odd primes. See Re-mark 6.5.

K–module structure map $K \otimes M \to M$ is in $\mathcal{U}$, and morphisms are K–module maps in $\mathcal{U}$.

Proposition 4.2. *Let $K \in \mathcal{K}$ be Noetherian, and $M \in K_{f.g.} - \mathcal{U}$. Then $L_0 K \in K_{f.g.} - \mathcal{U}$, and thus is Noetherian, and $L_0 M \in L_0 K_{f.g.} - \mathcal{U}$.*

See [HLS2, I(4.10)].

4.2. **The nilpotent filtration.** For $d \geq 0$, let $\mathcal{N}il_d \subset \mathcal{U}$ be the localizing subcategory generated by d–fold suspensions of unstable $\mathcal{A}$–modules. An unstable module M admits a natural filtration

$$\cdots \subseteq nil_2 M \subseteq nil_1 M \subseteq nil_0 M = M,$$

where $nil_d M$ is the largest submodule in $\mathcal{N}il_d$.

As observed in [K3, Prop.2.2], $nil_d M / nil_{d+1} M = \Sigma^d R_d M$, where $R_d M$ is a reduced unstable module. (See also [S2, Lemma 6.1.4].)

Proposition 4.3. *The functors $R_d : \mathcal{U} \to \mathcal{U}$ satisfy the following properties.*

(a) *There are a natural isomorphisms $R_*(M \otimes N) \simeq R_* M \otimes R_* N$ of graded objects in $\mathcal{U}$.*

(b) *There are natural isomorphism $T_V R_d M \simeq R_d T_V M$.*

(c) *Let $K \in \mathcal{K}$ be Noetherian, and $M \in K_{f.g.} - \mathcal{U}$. Then $R_0 K$ is also a Noetherian unstable algebra, and $R_d M \in R_0 K_{f.g.} - \mathcal{U}$, for all d.*

For the first two properties, see [K3, §3], and the last follows easily from the first.

Now let $\bar{R}_d M$ denote the nilclosure of $R_d(M)$. Thus $R_d M \subseteq L_0 R_d M = \bar{R}_d M$. We also note that $L_0 M = \bar{R}_0 M$.

The previous propositions combine to prove the following.

Proposition 4.4. *The functors $\bar{R}_d : \mathcal{U} \to \mathcal{U}$ satisfy the following properties.*

(a) *There are natural isomorphisms $\bar{R}_*(M \otimes N) \simeq \bar{R}_* M \otimes \bar{R}_* N$ of graded objects in $\mathcal{U}$.*

(b) *There are natural isomorphisms $T_V \bar{R}_d M \simeq \bar{R}_d T_V M$.*

(c) *Let $K \in \mathcal{K}$ be Noetherian, and $M \in K_{f.g.} - \mathcal{U}$. Then $\bar{R}_0 K$ is also a Noetherian unstable algebra, and $\bar{R}_d M \in \bar{R}_0 K_{f.g.} - \mathcal{U}$, for all d.*

Henn [H] proved the following important finiteness result.

Proposition 4.5. *Let $K \in \mathcal{K}$ be Noetherian, and $M \in K_{f.g.} - \mathcal{U}$. Then the nilpotent filtration of M has finite length. Equivalently, $\bar{R}_d M = 0$ for $d >> 0$.*

4.3. **Invariant rings as Noetherian unstable algebras.**

Proposition 4.6. *Let $K \in \mathcal{K}$ be Noetherian. Given any subgroup $G < Aut_{\mathcal{K}}(K)$, the invariant ring K^G is again a Noetherian unstable algebra, and K is a finitely generated K^G–module.*

Proof. Let $D(d) \in \mathcal{K}$ denote the dth Dickson algebra: in the notation of §3, $D(d)$ is the polynomial algebra $S^*((\mathbb{F}_p)^d)^{GL_d(\mathbb{F}_p)}$. Then let $D(d,j) \in \mathcal{K}$ denote the subalgebra consisting of p^j powers of the elements of $D(d)$. In [BoZ, Thm.A.1], Lannes shows that, if $d = \dim K$, then there is an embedding of unstable algebras $D(d,j) \rightarrow K$ that is unique in the sense that any two such embeddings will agree after restriction to $D(d,k)$ with k large enough. As G is necessarily finite, it follows that for large enough j, there is an embedding $D(d,j) \rightarrow K^G$ such that K is a finitely generated $D(d,j)$–module. The proposition follows. $\square$

Our various propositions apply to the case when $K = H^*(P)$.

Corollary 4.7. *Let P be a finite p–group.*

(a) *Both $H^*(P)^{Out(P)}$ and $\mathcal{H}^*(P)^{Out(P)}$ are Noetherian.*

(b) $e_S H^*(P) \in H^*(P)^{Out(P)}_{f.g.} - \mathcal{U}$ *and* $e_S \bar{R}_d H^*(P) \in \mathcal{H}^*(P)^{Out(P)}_{f.g.} - \mathcal{U}$, *for all irreducible $\mathbb{F}_p[Out(P)]$–modules S and all d.*

4.4. **How to deal with odd primes.** When p is odd, $K \in \mathcal{K}$ is not necessarily commutative unless K is concentrated in even degrees. To use standard definitions and results from the commutative algebra literature, and for other technical reasons, it is useful to have a systematic way of ridding ourselves of this problem.

A standard thing to do, done many times before, and going back at least to [LZ], goes as follows. Let $\mathcal{U}'$ denote the full subcategory of $\mathcal{U}$ consisting of modules concentrated in even degrees. Given $M \in \mathcal{U}$, we let $M' \in \mathcal{U}'$ denote the image of M under the right adjoint of the inclusion $\mathcal{U}' \subset \mathcal{U}$: in more down-to-earth terms, M' is the largest submodule of M contained in even degrees. It is easy to see that if K is an unstable algebra, then K' is a subalgebra. As an example, $H^*(V)' = S^*(V)$.

Statements about $K_{f.g.} - \mathcal{U}$ become statements about $K'_{f.g.} - \mathcal{U}$ using the next result.

Proposition 4.8. *If $K \in \mathcal{K}$ is Noetherian, so is K', and K is a finitely generated K'–module.*

This is [BrZ1, Lemma 5.2].

4.5. **Krull dimension.** We will generally work with the standard definition of Krull dimension. Given a commutative graded Noetherian $\mathbb{F}_p$–algebra K, $\dim K = d$ if

$$\wp_0 \subset \cdots \subset \wp_d$$

is a chain of prime ideals in K of maximal length. If M is a finitely generated K–module, $\dim M$ is then defined to be $\dim(K/Ann(M))$, where $Ann(M)$ is the annihilator ideal of M.

Given a Noetherian $K \in \mathcal{K}$ and $M \in K_{f.g.} - \mathcal{U}$, the proposition of the last subsection shows that $M \in K'_{f.g.} - \mathcal{U}$, and thus $\dim M$ will be a well defined finite natural number.

In this situation, the Poincaré series of M, $\sum_{i=0}^{\infty}(\dim_{\mathbb{F}_p} M^i)t^i$, will be a rational function, and $\dim M$ equals the order of the pole at $t = 1$. There is a third way of calculating $\dim K$: it is the number d such that there exist algebraically independent elements $k_1, \ldots, k_d \in K$ with K finitely generated over $\mathbb{F}_p[k_1, \ldots, k_d]$. See [CTVZ, §10.2] for a nice discussion of these and related facts.

Proposition 2.6 of the introduction is a special case of the following.

Proposition 4.9. *Let $K \in \mathcal{K}$ be Noetherian. Given $M \in K_{f.g.} - \mathcal{U}$,*

$$\dim M = \max_d \{\dim \bar{R}_d M\}.$$

Proof. As the modules $\Sigma^d R_d M$ are the composition factors associated to a finite filtration of M, standard properties of Krull dimension [Mat, (12.D)] imply that $\dim M = \max_d \{\dim R_d M\}$. Recalling that $\bar{R}_d = L_0 R_d$, the next proposition finishes the proof. $\square$

Proposition 4.10. *Given K and M as above, if M is reduced then*

$$\dim M = \dim L_0 M.$$

This follows from the next two lemmas.

Lemma 4.11. *$Ann(M)$ is a sub $\mathcal{A}$–module of K.*

Proof. We are claiming that, if $kx = 0$ for all $x \in M$, then, for all $a \in \mathcal{A}$ and $x \in M$, we have $(ak)x = 0$. Fixing $a \in \mathcal{A}$, assume by induction that $(a'k)y = 0$ for all $y \in M$ and $a' \in \mathcal{A}$ with $|a'| < |a|$. With $\Delta a = \sum a' \otimes a''$, we then have

$$0 = a(kx)$$
$$= (ak)x + \sum_{|a'| < |a|} (a'k)(a''x)$$
$$= (ak)x,$$

where the last equality uses the inductive hypothesis. $\square$

Lemma 4.12. *With M reduced, $Ann(M) = Ann(L_0 M)$.*

Proof.

$$Ann(M) = \{k \in K \mid \mathcal{A}k \otimes M \to M \text{ is } 0\}$$
$$= \{k \in K \mid \mathcal{A}k \otimes M \to L_0 M \text{ is } 0\}$$
$$= \{k \in K \mid \mathcal{A}k \otimes L_0 M \to L_0 M \text{ is } 0\}$$
$$= Ann(L_0 M).$$

Here the first and last equalities are consequences of the last lemma. The second is immediate since M is reduced, i.e. $M \to L_0 M$ is monic. Finally, the third equality follows from the universal property of nilclosed modules, as $\mathcal{A}k \otimes M \to \mathcal{A}k \otimes L_0 M$ is a $\mathcal{N}il_1$–isomorphism. $\square$

4.6. **Depth.** If M is a K–module, and $I \subset K$ is an ideal, the depth of M with respect to I is defined to be the maximal length l of an M–regular sequence in I: $r_1, \ldots, r_l \in I$ such that for each i between 1 and l, r_i is not a zero divisor on $M/(r_1, \ldots, r_{i-1})M$.

If K is a Noetherian unstable algebra, and $M \in K_{f.g.} - \mathcal{U}$, we let $\mathrm{depth}_K M$ denote the depth of M with respect to the ideal of positive degree elements in K.

It is standard that $\mathrm{depth}_K M \leq \dim N$, where $N \subset M$ is any nonzero submodule. This is usually stated in the following equivalent formulation [Mat, Thm.29]: $\mathrm{depth}_K M \leq \dim K/\wp$ where $\wp$ is any associated prime ideal, i.e. a prime ideal arising as the annihilator of an element of M. The associated primes include all the minimal primes in the support of M [Mat, Thm.9].

Quillen [Q2, Prop.11.2] shows that the minimal primes correspond to the maximal elementary abelian subgroups, and so $\text{depth}_{H^*(P)} H^*(P) \leq \text{mrk}(P)$, as asserted in the introduction.

The work of Bourguiba, Lannes, and Zarati [BoZ] shows

Proposition 4.13. *Let $K \in \mathcal{K}$ be Noetherian, $G < Aut_{\mathcal{K}}(K)$, and $M \in K_{f.g.} - \mathcal{U}$. Then $\text{depth}_{K^G} M = \text{depth}_K M$.*

The point here is that the main theorem of [BoZ] says that $\text{depth}_K M$ can be calculated by using a very specific regular sequence of 'generalized Dickson invariants', and, as discussed in the proof of Proposition 4.6, Lannes' theorem in the appendix of [BoZ] says that these elements will be in K^G.

Corollary 4.14. *Let $K \in \mathcal{K}$ be Noetherian, $G < Aut_{\mathcal{K}}(K)$, $M \in K_{f.g.} - \mathcal{U}$, and N be a nonzero direct summand of M, viewed as a K^G–module. Then $\dim N \geq \text{depth}_K M$.*

Proof. $\dim N \geq \text{depth}_{K^G} N \geq \text{depth}_{K^G} M = \text{depth}_K M$. $\square$

Specializing further, we have

Corollary 4.15. *Let P be a finite p–group, and S an irreducible $\mathbb{F}_p[Out(P)]$–module. If $e_S H^*(P) \neq 0$, then $\dim e_S H^*(P) \geq \text{depth}_{H^*(P)} H^*(P)$. If $e_S \bar{R}_d H^*(P) \neq 0$, then $\dim e_S \bar{R}_d H^*(P) \geq \text{depth}_{\mathcal{H}^*(P)} \bar{R}_d(P)$.*

Duflot showed [D] that $\text{depth}_{H^*(P)} H^*(P) \geq \text{rk}(C) = c(P)$. Subsequent proofs, beginning with [BrH], have emphasized that this theorem is a reflection of the $H^*(C)$–comodule structure on $H^*(P)$ induced by the multiplication homomorphism $C \times P \to P$. This same homomorphism also induces an $H^*(C)$–comodule structure on $\bar{R}_* H^*(P)$, and the proof of Duflot's theorem given in [CTVZ, Thm.12.3.3] goes through without change to prove

Proposition 4.16. $\text{depth}_{\mathcal{H}^*(P)} \bar{R}_*(P) \geq c(P)$.

This proposition and the preceeding corollary imply Proposition 2.7.

5. Proof of Theorem 2.1

In this section, we give a proof of Theorem 2.1 based on the identification of $\mathcal{U}/\mathcal{N}il$ as a certain functor category.

Following [HLS1], let $\mathcal{F}$ be the category of functors from finite dimensional $\mathbb{F}_p$–vector spaces to $\mathbb{F}_p$–vector spaces. This is an abelian category in the obvious way: $F \to G \to H$ is exact if $F(W) \to G(W) \to H(W)$ is exact for all W.

Let $l : \mathcal{U} \to \mathcal{F}$ be defined by $l(M)(W) = (T_W M)^0 = \text{Hom}_{\mathcal{U}}(M, H^*(W))'$. (Here M' denotes the continuous dual of a profinite vector space M.) This has right adjoint $r : \mathcal{F} \to \mathcal{U}$ given by $r(F)^d = \text{Hom}_{\mathcal{F}}(H_d, F)$, where $H_*(W)$ is the mod p homology of the group W.

Proposition 5.1. *The functors l and r satisfy the following properties.*

(a) *l is exact.*

(b) *The natural transformation $M \to r(l(M))$ identifies with $M \to L_0 M$. In particular $l(M) = 0$ if and only if $M \in \mathcal{N}il_1$.*

(c) *Both l and r commute with tensor products.*

For proofs of these properties see [HLS1, K2, S2].

The proposition implies that a nilclosed module, like $\mathcal{H}^*(P)$, is completely determined as an object in $\mathcal{U}$ by its associated functor. It is well known that Quillen's work allows for the identification of $l(H^*(P)) = l(\mathcal{H}^*(P))$. Let $\text{Rep}(W,P) = \text{Hom}(W,P)/\text{Inn}(P)$.

Proposition 5.2. $l(\mathcal{H}^*(P))(W) = \mathbb{F}_p^{\text{Rep}(W,P)}$.

For a simple proof, see [K1].

Proof of the first statement of Theorem 2.1. By the discussion above, an irreducible $\mathbb{F}_p[Out(P)]$–module S occurs in $\mathcal{H}^*(P)$ if and only if it appears in the permutation module $\mathbb{F}_p^{\text{Rep}(W,P)}$ for some W.

Let $\alpha : W \to P$ represent an element in $\text{Rep}(W,P)$. The stablizer of the $Out(P)$–orbit of this element will be $Out_0(P,V)$ where $V = \alpha(W)$. Thus, as an $Out(P)$–module, $\mathbb{F}_p^{\text{Rep}(W,P)}$ is a direct sum of modules of the form $\mathbb{F}_p^{Out(P)/Out_0(P,V)}$, and every such module appears as a summand of $\mathbb{F}_p^{\text{Rep}(W,P)}$ if $\text{rk}\, W \geq \text{rk}\, V$.

Finally, we note that if $V_1 < V_2 < P$, then $Out_0(P,V_2) \leq Out_0(P,V_1)$, and thus $\mathbb{F}_p^{Out(P)/Out_0(P,V_1)}$ is a quotient of $\mathbb{F}_p^{Out(P)/Out_0(P,V_2)}$. It follows that if S occurs in $\mathbb{F}_p^{Out(P)/Out_0(P,V)}$ for some $V < P$, then S occurs in $\mathbb{F}_p^{Out(P)/Out_0(P,V)}$ for some maximal $V < P$. $\qquad\square$

We now finish the proof of Theorem 2.1 by showing that if S occurs in the permutation module $\mathbb{F}_p^{Out(P)/Out_0(P,V)}$ then $\dim e_S \mathcal{H}^*(P) \geq \text{rk}\, V$. We do this by looking more carefully at $\mathbb{F}_p^{\text{Rep}(W,P)}$ as an object in $\mathcal{F}$ with $Out(P)$ action.

We begin by giving an increasing filtration of the set valued bifunctor $\text{Rep}(W,P)$. Let $\text{Rep}_k(W,P) \subseteq \text{Rep}(W,P)$ be the set of elements represented by homomorphisms with image of rank at most k. The inclusions

$$\{0\} = \text{Rep}_0(W,P) \subseteq \text{Rep}_1(W,P) \subseteq \cdots \subseteq \text{Rep}_{\text{rk}(P)}(W,P) = \text{Rep}(W,P)$$

induces epimorphisms of objects in $\mathcal{F}$ with $Out(P)$ action

$$\mathbb{F}_p = \mathbb{F}_p^{\text{Rep}_0(W,P)} \leftarrow \mathbb{F}_p^{\text{Rep}_1(W,P)} \leftarrow \cdots \leftarrow \mathbb{F}_p^{\text{Rep}_{\text{rk}(P)}(W,P)} = \mathbb{F}_p^{\text{Rep}(W,P)}.$$

An appeal to Proposition 4.10 thus shows

Lemma 5.3. $\dim e_S \mathcal{H}^*(P) = \max_k \{\dim e_S r(\ker\{\mathbb{F}_p^{\text{Rep}_k(W,P)} \to \mathbb{F}_p^{\text{Rep}_{k-1}(W,P)}\})\}$.

To more usefully describe $\ker\{\mathbb{F}_p^{\text{Rep}_k(W,P)} \to \mathbb{F}_p^{\text{Rep}_{k-1}(W,P)}\}$, we need to introduce some notation. Given $V < P$, let $GL_P(V) = Out(P,V)/Out_0(P,V)$. Note that $GL_P(V)$ is naturally a subgroup of $GL(V)$ and also acts on the right of the $Out(P)$–set $Out(P)/Out_0(P,V)$.

Lemma 5.4. *There is an isomorphism of objects in $\mathcal{F}$ with $Out(P)$ action,*

$$\ker\{\mathbb{F}_p^{\text{Rep}_k(W,P)} \to \mathbb{F}_p^{\text{Rep}_{k-1}(W,P)}\} \simeq \prod_{[V]} \mathbb{F}_p^{\text{Epi}(W,V) \times_{GL_P(V)} Out(P)/Out_0(P,V)},$$

where the product is over $Aut(P)$–orbits of elementary abelian subgroups of rank k.

Now we need to identify the nilclosed unstable module with $Out(P)$ action associated to $\mathbb{F}_p^{\mathrm{Epi}(W,V) \times_{GL_P(V)} Out(P)/Out_0(P,V)}$.

Let $c_V \in S^*(V)^{GL(V)}$ be the 'top' Dickson invariant: c_V is the product of all the elements $\beta(x) \in H^2(V)$, with $0 \neq x \in H^1(V)$. The key property we need is that $\eta^*(c_V) = 0$ for all proper inclusions $\eta : U < V$.

Lemma 5.5 (Compare with [HLS1, proof of Thm.II.6.4]). $l(c_V S^*(V))(W) = \mathbb{F}_p^{\mathrm{Epi}(W,V)}$.

Corollary 5.6. *The functor l assigns to the reduced module*

$$[c_V S^*(V) \otimes \mathbb{F}_p^{Out(P)/Out_0(P,V)}]^{GL_P(V)},$$

the object in $\mathcal{F}$ which sends W to $\mathbb{F}_p^{\mathrm{Epi}(W,V) \times_{GL_P(V)} Out(P)/Out_0(P,V)}$.

To finish the proof of Theorem 2.1, we are left needing to prove

Proposition 5.7. *If an irreducible $\mathbb{F}_p[Out(P)]$–module S occurs in the permutation module $\mathbb{F}_p[Out(P)/Out_0(P,V)]$, then*

$$\dim e_S[c_V S^*(V) \otimes \mathbb{F}_p^{Out(P)/Out_0(P,V)}]^{GL_P(V)} = \mathrm{rk}\, V.$$

Proof. The proof is similar to the proof of Conjecture A when $P = V$ given in §3. Our first observation is that $e_S[c_V S^*(V) \otimes \mathbb{F}_p^{Out(P)/Out_0(P,V)}]^{GL_P(V)}$ is a $S^*(V)^{GL(V)}$–submodule of the free $S^*(V)^{GL(V)}$–module $S^*(V) \otimes \mathbb{F}_p^{Out(P)/Out_0(P,V)}$. As $\dim S^*(V)^{GL(V)} = \mathrm{rk}\, V$, it suffices to show that if S occurs in the permutation module $\mathbb{F}_p[Out(P)/Out_0(P,V)]$, then S occurs in

$$[c_V S^*(V) \otimes \mathbb{F}_p^{Out(P)/Out_0(P,V)}]^{GL_P(V)}.$$

The group ring $\mathbb{F}_p[GL_P(V)]$ occurs as a submodule of $S^*(V)$ [Alp, p.45], and thus as a submodule of $c_V S^*(V)$. Thus S will occur in $[c_V S^*(V) \otimes \mathbb{F}_p^{Out(P)/Out_0(P,V)}]^{GL_P(V)}$ if it occurs in $[\mathbb{F}_p[GL_P(V)] \otimes \mathbb{F}_p^{Out(P)/Out_0(P,V)}]^{GL_P(V)}$. But this last $\mathbb{F}_p[Out(P)]$–module rewrites as $\mathbb{F}_p^{Out(P)/Out_0(P,V)}$. $\qquad\square$

6. Proof of Theorem 2.3 and related results

Theorem 2.1 tells us that calculating $\dim e_S \mathcal{H}^*(P)$ amounts to understanding the $\mathbb{F}_p[Out(P)]$–module composition factors of the permutation modules

$$\mathbb{F}_p[Out(P)/Out_0(P,V)],$$

when $V < P$ is a maximal elementary abelian subgroup.

In this section, we use group theory to find various conditions ensuring that all irreducibles occur in this way.

6.1. **Composition factors of permutation modules.** We begin with an elementary, but useful, lemma about permutation modules.

Lemma 6.1. *Let H be a subgroup of a finite group G. Then* **(a)** $\Rightarrow$ **(b)** $\Rightarrow$ **(c)**, *and, if H is also normal,* **(c)** $\Rightarrow$ **(a)**.

(a) *H is a p–group.*

(b) *Every irreducible $\mathbb{F}_p[G]$–module occurs as a submodule of $\mathbb{F}_p[G/H]$.*

(c) *Every irreducible $\mathbb{F}_p[G]$–module occurs as a composition factor of $\mathbb{F}_p[G/H]$.*

Proof. We begin by reminding the reader that permutation modules are self dual, so an irreducible S occurs as a submodule of $\mathbb{F}_p[G/H]$ if and only if S occurs as a quotient module. This happens if and only if there exists a nonzero homomorphism $\mathbb{F}_p[G/H] \to S$, which is equivalent to $S^H \neq 0$.

If H is a p–group, then $M^H \neq 0$ for all nonzero $\mathbb{F}_p[H]$–modules M. Thus (a) implies (b), and (b) implies (c) is obvious.

Now suppose that H is normal. Then the composition factors of $\mathbb{F}_p[G/H]$ are just the irreducible $\mathbb{F}_p[G/H]$–modules pulled back to G. But if H is not a p–group, then G/H has fewer p' conjugacy classes than G, and thus fewer irreducible modules. Thus (c) implies (a) under the normality assumption. $\qquad\square$

Remark 6.2. It is not hard to see that, if H is not normal, then neither implication (b) $\Rightarrow$ (a) nor (c) $\Rightarrow$ (b) need hold. For the former, let $p = 3$, $G = SL_2(\mathbb{F}_3)$, and $H = \mathbb{Z}/2$ permuting a basis for $(\mathbb{F}_3)^2$. For the latter, let $p = 3$, $G = \Sigma_3$, and $H = \mathbb{Z}/2$.

Corollary 6.3. *If a p–group P has a unique maximal elementary abelian subgroup V, then every irreducible occurs in $\mathbb{F}_p[Out(P)/Out_0(P,V)]$ if and only if $Out_0(V)$ is a p–group.*

Proof. The hypothesis implies that $Out_0(P,V)$ arises as the kernel of an evident homomorphism $Out(P) \to Aut(V)$, and thus is normal in $Out(P)$. Now the lemma applies. $\qquad\square$

6.2. **p'–automorphisms of p–groups.** We will use various well known results about detecting automorphisms of a p–group P of order prime to p.

The most classic is due to Burnside and Hall, and goes as follows. Let $\Phi(P)$ be the Frattini subgroup of P, so that $P/\Phi(P) = H_1(P; \mathbb{F}_p)$.

Proposition 6.4 ([Gor, Thm.5.1.4]). *The kernel of the homomorphism $Aut(P) \to Aut(P/\Phi(P))$ is a p–group.*

With groups A and B as indicated, the Thompson $A \times B$ lemma ([Asch, (24.2)], [Gor, Thm.5.3.4]) immediately applies to prove the next proposition.

Proposition 6.5. *Let B be an abelian subgroup of a finite p–group P, and let $A = \{\alpha \in Aut(P) \mid \alpha(g) = g \text{ for all } g \in C_P(B)\}$. Then A is a p–group.*

Corollary 6.6. *For all elementary abelian $V < P$, the kernel of the homomorphism $Aut(P,V) \to Aut(C_P(V))$ is a p–group. Thus if $Aut(C_P(V))$ is a p–group so is $Aut(P,V)$, and if $Aut_0(C_P(V),V)$ is a p–group so is $Aut_0(P,V)$.*

The next result requires that p be odd. Let $\Omega_1(P)$ be the subgroup of P generated by the elements of order p.

Proposition 6.7 ([Gor, Thm.5.3.10]). *Let P be a p–group, with p odd. Then the kernel of the homomorphism $Aut(P) \to Aut(\Omega_1(P))$ is a p–group.*

Corollary 6.8. *Let Q is p–central, with p odd, and with maximal central elementary abelian subgroup C. Then $Aut_0(Q,C)$ is a p–group.*

This follows from the proposition, noting that $C = \Omega_1(Q)$, since Q is p–central.

Proof of Theorem 2.3. The first statement follows immediately from Corollary 6.6. Now we note that, if $V < P$ is a maximal elementary abelian subgroup, then $C_P(V)$ is p–central with V as its maximal central elementary abelian subgroup. Thus Corollary 6.8 applies to prove the second statement. $\square$

6.3. 2–**central group examples.** We elaborate on Example 2.4.

Example 6.9. Let G_t be the 2–Sylow subgroup of $SU_3(\mathbb{F}_{2^t})$. This is a 2–central group of rank t, and if C_t is its maximal elementary abelian subgroup, $Out_0(G_t, C_t)$ contains a cyclic subgroup of order $2^t + 1$, and thus is not a 2–group.

This will be a consequence of a few facts[4] about G_t.

Let $q = 2^t$. Given $a \in \mathbb{F}_{q^2}$, let $\bar{a} = a^q$, so that $\mathbb{F}_q$ identifies with the set of $b \in \mathbb{F}_{q^2}$ such that $b + \bar{b} = 0$. Then $G_t \subset SU_3(\mathbb{F}_q) \subset GL_3(\mathbb{F}_{q^2})$ can be described as

$$G_t = \left\{ A(a, b) = \begin{pmatrix} 1 & a & b \\ 0 & 1 & \bar{a} \\ 0 & 0 & 1 \end{pmatrix} \; : \; a, b \in \mathbb{F}_{q^2} \text{ with } b + \bar{b} = a\bar{a} \right\}.$$

The center C_t of G_t is the set of matrices of the form $A(0, b)$: note that then $b \in \mathbb{F}_q$. All other elements have order 4, and thus G_t is 2–central sitting in the central extension

$$C_t \to G_t \to G_t/C_t,$$

with $C_t \simeq \mathbb{F}_q$ and $G_t/C_t \simeq \mathbb{F}_{q^2}$.

The normalizer of G_t in $SU_3(\mathbb{F}_{2^t})$ is the semidirect product $G_t \rtimes T_t$, where T_t is the set of matrices

$$\left\{ D(c) = \begin{pmatrix} c & 0 & 0 \\ 0 & \bar{c}c^{-1} & 0 \\ 0 & 0 & \bar{c}^{-1} \end{pmatrix} \; : \; c \in \mathbb{F}_{q^2}^\times \right\}.$$

Direct computation shows that

$$D(c)A(a, b)D(c^{-1}) = A(\bar{c}^{-1}c^2 a, \bar{c}cb) = A(c^{2-q}a, c^{q+1}b).$$

From this, one deduces that $T_t \to Aut(G_t/C_t)$, and thus $T_t \to Aut(G_t)$, is monic if $\gcd(q - 2, q^2 - 1) = 1$. This is the case unless $q = 2$, i.e., $t = 1$. Meanwhile, the kernel of the homomorphism $T_t \to Aut(C_t)$ identifies with the kernel of the multiplicative norm $\mathbb{F}_{q^2}^\times \to \mathbb{F}_q^\times$, and thus is cyclic of order $q + 1$. Thus, if $t \geq 2$, $Aut_0(G_t, C_t) = \ker\{Aut(G_t) \to Aut(C_t)\}$ contains a cyclic group of order $(q + 1)$.

(When $t = 1$, $G_t = Q_8$, and so $Aut_0(G_t, C_t)$ also contains a group of order $(q + 1) = 3$.)

Remark 6.10. The group G_2 has been of interest to those studying group cohomology. Even though it is 2–central, a presentation of its cohomology ring is remarkably nasty to write down: see the calculations for group number # 187 of order 64 in [Ca, CTVZ]. It is the smallest group with nontrivial products in its essential cohomology [Gr]. The calculation of the nilpotent length of p–central groups in [K4, K5] shows that its nilpotent length is 14, which seems likely to be maximal among all groups of order 64. Presumably, the groups G_t for larger t are similarly interesting.

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[4]The authors are very grateful to David Green for showing us his unpublished 'Appendix B' to [Gr], which is our source of information about these groups.

6.4. p–**groups with reduced cohomology.** Recall that P has reduced cohomology if and only if $H^*(P)$ is detected by restriction to the various $V < P$, i.e., $H^*(P) \to \mathcal{H}^*(P)$ is monic.

We restate Proposition 2.5.

Proposition 6.11. *If $H^*(P)$ is reduced, and $V < P$ is a maximal elementary abelian subgroup, then $C_P(V) = V$.*

Recalling that $C_P(V)$ is p–central if $V < P$ is maximal, the proposition is a consequence of the next two lemmas.

Lemma 6.12. *If $H^*(P)$ is reduced, so is $H^*(C_P(V))$ for any elementary abelian $V < P$.*

Lemma 6.13. *If Q is a p–central p–group with reduced cohomology, then Q is elementary abelian.*

Proof of Lemma 6.12. Properties of the functor T_V imply that if $M \in \mathcal{U}$ is reduced so is $T_V M$: recalling that T_V preserves monomorphisms, property (b) of Proposition 4.1 impies this. Now one uses that, given $V < P$, $H^*(C_P(V))$ is a (canonical) direct summand in $T_V H^*(P)$. (See [HLS2, I(5.2)] for a calculation of $T_V H^*(P)$.) $\qquad\square$

Proof of Lemma 6.13. Let C be the center of Q. The hypothesis is that $H^*(Q) \to H^*(C)$ is monic, which implies that the inflation map $H^*(Q/C) \to H^*(Q)$ is zero. But $H^1(Q/C) \to H^1(Q)$ is always monic, so we conclude that $H^1(Q/C) = 0$. But, since Q/C is a p–group, this means that Q/C is trivial, i.e., $C = Q$. $\qquad\square$

6.5. **A remark about odd prime p–central groups.** Using a result of Henn and Priddy, Theorem 2.3 has an addendum. For odd prime p–central groups P, the refinement of Conjecture A described in Remark 3.1 holds: indeed, $\dim e\mathcal{H}^*(P) = \mathrm{rk}(P)$ for all idempotents $e \in A(P, P) \otimes \mathbb{F}_p$ that project to a nonzero element under the retraction of algebras $A(P, P) \otimes \mathbb{F}_p \to \mathbb{F}_p[Out(P)]$. Here dimension is defined via the Poincaré series, and we remind the reader that $A(P, P)$ is the double Burnside ring.

The argument goes as follows. Given an irreducible $\mathbb{F}_p[Out]$–module S, let $e_S \in \mathbb{F}_p[Out(P)]$ and $\tilde{e}_S \in A(P, P) \otimes \mathbb{F}_p$ be the associated primitive idempotents[5]. Then $e_S \mathcal{H}^*(P) = \tilde{e}_S \mathcal{H}^*(P) \oplus M^*$ where M^* is a finite direct sum of modules of the form $\tilde{e}_T \mathcal{H} * (Q)$ for appropriate pairs (Q, T). By [HP, Prop.1.6.1 and Lem.2.1], if P is an odd prime p–central group, the only Q's that can occur here will be proper retracts of P. But such groups will necessarily have strictly smaller ranks. Thus $\dim M^* < \mathrm{rk}(P)$, and so $\dim \tilde{e}_S \mathcal{H}^*(P) = \dim e_S \mathcal{H}^*(P) = \mathrm{rk}(P)$.

7. Proof of Theorem 2.8

7.1. **A formula for $\bar{R}_d H^*(P)$.** Recall that $\bar{R}_0 H^*(P) = \lim_{V \in \mathcal{A}^C(P)} H^*(V)$. Using the fact that morphisms in $\mathcal{A}(V)$ factor as inner automorphisms composed with inclusions, this rewrites as follows: there is a natural isomophism

$$\bar{R}_0 H^*(P) = \mathrm{Eq}\left\{ \left[\prod_V H^*(V) \right]^{Inn(P)} \overset{\mu}{\underset{\nu}{\rightrightarrows}} \prod_{V_1 < V_2} H^*(V_1) \right\},$$

---

[5]Thus $\tilde{e}_S$ projects to e_S, and, in the terminology of [MP], $e_S BP$ is the super dominant summand associated to the dominant summand $\tilde{e}_S BP$.

where the products are over objects and inclusions in $\mathcal{A}^C(P)$, and μ and ν are induced by $1 : H^*(V_1) \to H^*(V_1)$ and $\eta^* : H^*(V_2) \to H^*(V_1)$ for each inclusion $\eta : V_1 < V_2$ in $\mathcal{A}^C(P)$.

We begin this section with a formula for $\bar{R}_d H^*(P)$ that generalizes this.

The ingredients of our formula are the following.

First, recall our notation from the introduction: if W is a central elementary abelian p–subgroup of Q, then $P_W H^*(Q)$ denotes the primitives in the $H^*(W)$–comodule $H^*(Q)$.

An automorphism $\alpha : P \to P$ induces an isomorphism of unstable algebras $P_{\alpha(V)} H^*(C_P(\alpha(V))) \to P_V H^*(C_P(V))$. Thus $P_V H^*(C_P(V))$ is an $Aut(P, V)$–module, and the full automorphism group $Aut(P)$ acts on the product

$$\prod_{V \in \mathcal{A}^C(P)} H^*(V) \otimes P_V H^d(C_P(V)).$$

Similarly, an inclusion $\eta : V_1 < V_2$ induces an inclusion $C_P(V_2) < C_P(V_1)$, and thus a map of unstable algebras $\eta_* : P_{V_1} H^*(C_P(V_1)) \to P_{V_1} H^*(C_P(V_2))$. We will also denote by $i : P_{V_2} H^*(C_P(V_2)) \to P_{V_1} H^*(C_P(V_2))$ the evident inclusion of primitives.

Starting from formulae in [HLS2], in [K5], the author will deduce the following formula for $\bar{R}_d H^*(P)$.

Proposition 7.1. $\bar{R}_d H^*(P)$ *is naturally isomorphic to the equalizer*

$$Eq\left\{ \left[\prod_V H^*(V) \otimes P_V H^d(C_P(V)) \right]^{Inn(P)} \xrightarrow[\nu]{\mu} \prod_{V_1 < V_2} H^*(V_1) \otimes P_{V_1} H^d(C_P(V_2)) \right\},$$

where the products are over objects and proper inclusions in $\mathcal{A}^C(P)$, and μ and ν are induced by

$$1 \otimes \eta_* : H^*(V_1) \otimes P_{V_1} H^d(C_P(V_1)) \to H^*(V_1) \otimes P_{V_1} H^d(C_P(V_2))$$

and

$$\eta^* \otimes i : H^*(V_2) \otimes P_{V_2} H^d(C_P(V_2)) \to H^*(V_1) \otimes P_{V_1} H^d(C_P(V_2))$$

for each proper inclusion $\eta : V_1 < V_2$ in $\mathcal{A}^C(P)$.

Otherwise said, $x = (x_V) \in [\prod_V H^*(V) \otimes P_V H^d(C_P(V))]^{Inn(P)}$ is in $\bar{R}_d H^*(P)$ exactly when the components are related by $(1 \otimes \eta_*)(x_{V_1}) = (\eta^* \otimes i)(x_{V_2})$, for each proper inclusion $\eta : V_1 < V_2$ in $\mathcal{A}^C(P)$.

7.2. $\bar{R}_d H^*(Q)$ when Q is p–central. The formula for $\bar{R}_d H^*(Q)$ simplies dramatically if Q is p–central, as the first product in the equalizer formula is indexed by one element, and the second is empty.

Proposition 7.2. *Suppose Q is p–central with maximal central elementary abelian subgroup C. Then*

$$\bar{R}_d H^*(Q) \simeq H^*(C) \otimes P_C H^d(Q).$$

As a consequence we can prove Proposition 2.10, which said that if P is 2–central, then every irreducible $\mathbb{F}_2[Out_0(Q, C)]$–module occurs as a composition factor in $P_C H^*(Q)$.

Proof of Proposition 2.10. Every irreducible $\mathbb{F}_2[Out(Q)]$–module occurs in $H^*(Q)$, so the same is true for irreducible $\mathbb{F}_2[Out_0(Q,C)]$–modules. Thus every irreducible $\mathbb{F}_2[Out_0(Q,C)]$–module occurs in $\bar{R}_*H^*(Q) \simeq H^*(C) \otimes P_C H^*(Q)$. But, by definition, $H^*(C)$ is a trivial $\mathbb{F}_2[Out_0(Q,C)]$–module, so the irreducibles must occur in the other factor, $P_C H^*(Q)$. $\hspace{1cm}\square$

Remark 7.3. In [K5], we will prove that if Q is p–central, the finite unstable algebra $P_C H^*(Q) \subset H^*(Q)$ identifies with the subalgebra of locally finite elements: $x \in H^*(Q)$ such that $\mathcal{A}x \subset H^*(Q)$ is finite.

7.3. **Proof of Theorem 2.8.** It will be useful to let $M^*(V)$ denote $P_V H^*(C_P(V))$. Similarly, given $V_1 < V_2$, let $M^*(V_2, V_1) = P_{V_1} H^*(C_P(V_2))$.

Given $V < P$, define ρ_V to be the composite

$$\bar{R}_d H^*(P) \subseteq [\prod_U H^*(U) \otimes M^d(U)]^{Inn(P)} \longrightarrow [H^*(V) \otimes M^d(V)]^{W_P(V)},$$

where the first map is the inclusion of $\mathbb{F}_p[Out(P)]$–modules given in Proposition 7.1, and the second is the canonical projection of $\mathbb{F}_p[Out(P,V)]$–modules. Then let

$$\tilde{\rho}_V : \bar{R}_d H^*(P) \to \mathrm{Ind}_{Out(P,V)}^{Out(P)}([H^*(V) \otimes M^d(V)]^{W_P(V)})$$

to be the $\mathbb{F}_p[Out(P)]$–module map adjoint to the $\mathbb{F}_p[Out(P,V)]$–module map ρ_V.[6]

Definition 7.4. Given an irreducible $\mathbb{F}_p[Out(P)]$–module S, and $d \geq 0$, define

$$\sup_d(S) = \{V \in \mathrm{ob}\ \mathcal{A}^C(P) \mid S \text{ occurs in } \tilde{\rho}_V(\bar{R}_d H^*(P))\}.$$

It is easy to check that $V \in \sup_d(S)$ if and only if $\rho_{\alpha(V)}(e_S \bar{R}_d H^*(P)) \neq 0$ for some $\alpha \in Aut(P)$. Equivalently, $V \in \sup_d(S)$ if and only if there exists $x = (x_U) \in e_S \bar{R}_d H^*(P)$ with $x_{\alpha(V)} \neq 0$ for some $\alpha \in Aut(P)$.

We will show that $\sup_d(S)$ has the properties described in Theorem 2.8.

Clearly, given $\alpha \in Aut(P)$, $V \in \sup_d(S)$ implies that $\alpha(V) \in \sup_d(S)$. Thus $\sup_d(S)$ is a union of $Aut(P)$ orbits in $\mathrm{ob}\,\mathcal{A}^C(P)$, i.e., property (a) of Theorem 2.8 holds.

Also evident is the first statement of property (c): $\sup_d(S) \neq \emptyset$ if and only if $e_S \bar{R}_d H^*(P) \neq 0$.

Not much harder is the proof of property (b), which we restate here as a lemma.

Lemma 7.5. *Given* $\eta : V_1 < V_2$, *suppose that* $\eta_* : M^d(V_1) \to M^d(V_2, V_1)$ *is monic. Then* $V_1 \in \sup_d(S)$ *implies that* $V_2 \in \sup_d(S)$.

Proof. Suppose $x = (x_U) \in e_S \bar{R}_d H^*(P)$ has $x_{\alpha(V_1)} \neq 0$, with $\alpha \in Aut(P)$. Let $\alpha(\eta) : \alpha(V_1) < \alpha(V_2)$ be the inclusion. Then η_* monic implies that $\alpha(\eta)_*$ is also monic. Since $(\alpha(\eta)^* \otimes i)(x_{\alpha(V_2)}) = (1 \otimes \alpha(\eta)_*)(x_{\alpha(V_1)})$, it follows that $x_{\alpha(V_2)} \neq 0$. $\hspace{1cm}\square$

Now suppose $e_S \bar{R}_d H^*(P) \neq 0$. As $e_S \bar{R}_d H^*(P)$ embeds in $\prod_V \rho_V(e_S \bar{R}_d H^*(P))$,

$$\dim e_S \bar{R}_d H^*(P) = \max\{\dim \rho_V(e_S \bar{R}_d H^*(P)) \mid V \in \sup_d(S)\}.$$

Furthermore, $\dim \tilde{\rho}_V(e_S \bar{R}_d H^*(P)) = \max\{\dim \rho_{\alpha(V)}(e_S \bar{R}_d H^*(P)) \mid \alpha \in Aut(P)\}$, and there is an obvious bound: $\dim \rho_V(e_S \bar{R}_d H^*(P)) \leq \dim H^*(V) = \mathrm{rk}(V)$. Thus the next two lemmas prove properties (c) and (d) of Theorem 2.8.

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[6]Recall that if H is a subgroup of a finite group G, induction Ind_H^G is both left and right adjoint to restriction Res_H^G.

Lemma 7.6. *If V is maximal in $\sup_d(S)$, then $\dim \tilde{\rho}_V(e_S \bar{R}_d H^*(P)) \geq \mathrm{rk}(V)$.*

Lemma 7.7. *If V is maximal in $\mathrm{ob}\,\mathcal{A}^C(P)$, then S occurs in $\tilde{\rho}_V(\bar{R}_d H^*(P))$ if and only if S occurs in $\mathrm{Ind}_{Out(P,V)}^{Out(P)}([\mathbb{F}_p[Aut(P,V)/Aut_0(P,V)] \otimes M^d(V)]^{W_P(V)})$.*

To prove these, we need to introduce yet more notation.

Given $V < P$, let $\mathrm{Ess}^*(V) = \ker\{M^*(V) \to \prod_{V<U} M^*(U)\}$, where the product is over all proper inclusions. Then $\alpha_*(\mathrm{Ess}^*(V)) = \mathrm{Ess}^*(\alpha(V))$ for all $\alpha \in Aut(P)$. In particular, $\mathrm{Ess}^*(V)$ is an $\mathbb{F}_p[Aut(P,V)]$–submodule of $M^*(V)$.

Lemma 7.8. *There is a map of $Out(P)$–modules*

$$\tilde{f}_V : \mathrm{Ind}_{Out(P,V)}^{Out(P)}([c_V H^*(V) \otimes \mathrm{Ess}^d(V)]^{W_P(V)}) \to \bar{R}_d H^*(P)$$

such that $\tilde{\rho}_V \circ \tilde{f}_V$ is the canonical inclusion induced by

$$c_V H^*(V) \otimes \mathrm{Ess}^d(V) \subseteq H^*(V) \otimes M^*(V).$$

Proof. It is equivalent to define an $Out(P,V)$–module map

$$f_V : [c_V H^*(V) \otimes \mathrm{Ess}^d(V)]^{W_P(V)} \to \bar{R}_d H^*(P)$$

such that $\tilde{\rho}_V \circ f_V$ is an appropriate inclusion. Given $x \in [c_V H^*(V) \otimes \mathrm{Ess}^d(V)]^{W_P(V)}$ define $f_V(x) \in \left[\prod_U H^*(U) \otimes M^d(U)\right]^{Inn(P)}$ by letting its Uth component, $f_V(x)_U$, be $(\alpha^{-1*} \otimes \alpha_*)(x)$ if $U = \alpha(V)$ with $\alpha \in Inn(P)$, and 0 otherwise.

We claim that $f_V(x)$ is, in fact, in $\bar{R}_d H^*(P)$. For given $\eta : V_1 < V_2$ with V_1 conjugate to V, $(1 \otimes \eta_*)(f_V(x)_{V_1}) = 0$ because $\eta_*(Ess^d(V)) = 0$. Similarly, given $\eta : V_1 < V_2$ with V_2 conjugate to V, $(\eta^* \otimes i)(f_V(x)_{V_2}) = 0$ because $\eta^*(c_V) = 0$.

Finally, checking that $\tilde{\rho}_V \circ f_V$ is what it should be is straightforward. $\qquad\square$

Next we observe that $\mathrm{Ind}_{Out(P,V)}^{Out(P)}([H^*(V) \otimes M^d(V)]^{W_P(V)})$ is a $H^*(V)^{GL(V)}$–module with the property that each nonzero cyclic $S^*(V)^{GL(V)}$–submodule is free.

Proof of Lemma 7.6. If V is maximal in $\sup_d(S)$, then, after possibly replacing V by $\alpha(V)$ for some $\alpha \in Aut(P)$, there exists $x \in e_S \bar{R}_s H^*(V)$ with $x_V \neq 0$ in $[H^*(V) \otimes M^d(V)]^{W_P(V)}$, and V maximal with this property. Reasoning as in the proof of Lemma 7.5, it follows that $x_V \in [H^*(V) \otimes \mathrm{Ess}^d(V)]^{W_P(V)}$. By the proof of Lemma 7.8,

$$c_V S^*(V)^{GL(V)} \tilde{\rho}_V(x) \subset \tilde{\rho}_V(\bar{R}_d H^*(P)).$$

By the observation above, we also have

$$c_V S^*(V)^{GL(V)} \tilde{\rho}_V(x) \subset e_S \mathrm{Ind}_{Out(P,V)}^{Out(P)}([H^*(V) \otimes M^d(V)]^{W_P(V)}).$$

Thus $c_V S^*(V)^{GL(V)} \tilde{\rho}_V(x)$ is contained in the intersection of these, i.e.

$$c_V S^*(V)^{GL(V)} \tilde{\rho}_V(x) \subset \tilde{\rho}_V(e_S \bar{R}_d H^*(P)).$$

But, as a graded $\mathbb{F}_p$–vector space, $c_V S^*(V)^{GL(V)} \tilde{\rho}_V(x)$ is just the $(|c_V| + |x|)^{th}$ suspension of $S^*(V)^{GL(V)}$, and so

$$\mathrm{rk}(V) = \dim c_V S^*(V)^{GL(V)} \tilde{\rho}_V(x) \leq \dim \tilde{\rho}_V(e_S \bar{R}_d H^*(P)).$$

$\qquad\square$

Proof of Lemma 7.7. If V is maximal in $\mathcal{A}^C(P)$, then the product defining $\text{Ess}^*(V)$ is empty, so $\text{Ess}^*(V) = M^*(V)$. From Lemma 7.8, we thus deduce that we have inclusions of $\mathbb{F}_p[Out(P)]$–modules

$$\text{Ind}_{Out(P,V)}^{Out(P)}([c_V H^*(V) \otimes M^d(V)]^{W_P(V)}) \subset \tilde{\rho}_V(\bar{R}_d H^*(P))$$

and

$$\tilde{\rho}_V(\bar{R}_d H^*(P)) \subset \text{Ind}_{Out(P,V)}^{Out(P)}([H^*(V) \otimes M^d(V)]^{W_P(V)}).$$

Thus the next lemma will finish the proof. $\qquad\square$

Lemma 7.9. *Let S be an irreducible $\mathbb{F}_p[Out(P)]$–module, and M be an $\mathbb{F}_p[Aut(P,V)]$–module. Then the following are equivalent.*

(a) *S occurs in $\text{Ind}_{Out(P,V)}^{Out(P)}([c_V H^*(V) \otimes M]^{W_P(V)})$.*

(b) *S occurs in $\text{Ind}_{Out(P,V)}^{Out(P)}([H^*(V) \otimes M]^{W_P(V)})$.*

(c) *S occurs in $\text{Ind}_{Out(P,V)}^{Out(P)}([\mathbb{F}_p[Aut(P,V)/Aut_0(P,V)] \otimes M]^{W_P(V)})$.*

Proof. Noting once again that multiplication by c_V is a monomorphism on the module in (b), we see that (a) and (b) are equivalent.

To simplify notation, let $G = Aut(P,V)/Aut_0(P,V)$ and $W = W_P(V)$. Thus $W < G < GL(V)$. The equivalence of (b) and (c) follows immediately from the following claim: if T is an irreducible $\mathbb{F}_p[G/W]$–module, then T occurs in $(H^*(V) \otimes M)^W$ if and only if T occurs in $(\mathbb{F}_p[G] \otimes M)^W$. The claim is a consequence of the well known fact that, for any $G < GL(V)$, every finite $\mathbb{F}_p[G]$–module embeds in $S^*(V)$ [Alp], and thus also in $H^*(V)$. Thus T occurs in $(H^*(V) \otimes M)^W$ if and only if T occurs in $(N \otimes M)^W$ for some finite $\mathbb{F}_p[G]$–module N, and this happens if and only if T occurs in (a direct sum of copies of) the module $(\mathbb{F}_p[G] \otimes M)^W$. $\qquad\square$

8. Conjecture C implies Conjecture A

If H is a subgroup of G, and M is an $\mathbb{F}_p[H]$–module, it is easy to verify the following: if every irreducible $\mathbb{F}_p[H]$–module occurs in M then every irreducible $\mathbb{F}_p[G]$–module occurs in $\text{Ind}_H^G M$. In light of Corollary 2.9, Conjecture A would thus be implied by

8.1. If $V < P$ is maximal, then every irreducible $\mathbb{F}_p[Out(P,V)]$–module occurs in $(\mathbb{F}_p[Aut(P,V)/Aut_0(P,V)] \otimes P_V H^*(C_P(V)))^{W_P(V)}$.

The $\mathbb{F}_p[Out(P,V)]$–module $(\mathbb{F}_p[Aut(P,V)/Aut_0(P,V)] \otimes P_V H^*(C_P(V)))^{W_P(V)}$ contains $\mathbb{F}_p[Aut(P,V)/Aut_0(P,V)]^{W_P(V)} \otimes P_V H^*(C_P(V))^{W_P(V)}$ as a submodule, and this submodule rewrites as

$$\text{Ind}_{Out_0(P,V)}^{Out(P,V)}(P_V H^*(C_P(V))^{W_P(V)}).$$

Conjecture B is the statement that every irreducible $\mathbb{F}_p[Out_0(P,V)]$–module occurs in $P_V H^*(C_P(V))^{W_P(V)}$. Thus Conjecture B would imply 8.1, and we have proved

Lemma 8.2. *If a pair (P,V) satisfies Conjecture B, then P satisfies Conjecture A.*

Now we prove that Conjecture C implies Conjecture B, establishing Theorem 2.11. More precisely, let $V < P$ be maximal, and assume that every irreducible $\mathbb{F}_2[Out_0(C_P(V), V)]$–module occurs as a submodule in $P_V H^*(C_P(V))$. We will show that then every irreducible $\mathbb{F}_2[Out_0(P, V)]$–module occurs as a composition factor in $P_V H^*(C_P(V))^{W_P(V)}$.

The key here is again to use Corollary 6.6, our application of the Thompson $A \times B$ lemma, which tells us that the kernel of $Aut_0(P, V) \to Aut_0(C_P(V), V)$ is a 2–group. Thus our assumption lets us conclude that every irreducible $\mathbb{F}_2[Aut_0(P, V)]$–module occurs as a submodule in $P_V H^*(C_P(V))$.

Now let S be an irreducible $\mathbb{F}_2[Out_0(P, V)]$–module. Equivalently, S can be regarded as an irreducible $\mathbb{F}_2[Aut_0(P, V)]$–module fixed by the subgroup $W_P(V)$. From our remarks in the last paragraph, there exists an inclusion $S \subset P_V H^*(C_P(V))$ of $\mathbb{F}_2[Aut_0(P, V)]$–modules, and thus an inclusion

$$S = S^{W_P(V)} \subset P_V H^*(C_P(V))^{W_P(V)}$$

of $\mathbb{F}_2[Out_0(P, V)]$–modules.

9. Some 2–central groups satisfying Conjecture C

If P is a 2–central group, then $C = \Omega_1(P) < P$ is the unique maximal elementary subgroup and is central. In this case, we write $Out_0(P)$ for $Out_0(P, C)$.

Recall that P satisfies Conjecture C if every irreducible $\mathbb{F}_2[Out_0(P)]$–module occurs as a submodule of the algebra of primitive $P_C H^*(P)$. In this section we verify this conjecture for many 2–central groups.

Our strategy is the following. First, we verify the conjecture for various indecomposable 2–central groups, including all of order up to 64. Second, we find various side hypotheses on a pair (P, Q) ensuring that if P and Q satisfy Conjecture C, so does $P \times Q$.

All calculations have been done by hand, using information listed on the website [Ca] or the book [CTVZ]. For starters, we note that a 2–group P is 2–central exactly when $\mathrm{rk}(Z(P)) = \mathrm{rk}(P)$, so 2–central groups can be easily identified. We occasionally use notation like '32#10' to denote a 2–group: this would be the group of order 32 and numbered 10 on [Ca] or in [CTVZ].

9.1. Indecomposable 2–central groups of order dividing 64. Let P be 2–central. Recall that $P_C H^0(P) = \mathbb{F}_2$, the trivial module, so P rather trivially satisfies Conjecture C whenever $Out_0(P)$ is a 2–group. This is often the case.

Lemma 9.1. *Let P be an indecomposable 2–central groups P of order dividing 64. Then $Out_0(P)$ is a 2–group unless P is one of the three groups 8#5 (Q_8), 64#162, or 64#187 (G_2 as in Example 2.4).*

We sketch how one checks this. Firstly, here is a table of the indecomposable 2–central groups P having *any* nontrivial automorphisms of order prime to 2, and the order of $Out(P)$ modulo its 2–Sylow subgroup.

P	$2'$ part of $|Out(P)|$
8#5	3
32#18	3
64#30	3
64#82	3
64#93	3
64#145	3
64#153	21
64#162	3
64#187	15

We note that 8#5 is Q_8, 64#153 is the 2–Sylow subgroup of the Suzuki group $Sz(8)$, and 64#187 is the 2–Sylow subgroup of $SU_3(\mathbb{F}_4)$ as in Example 2.4.

Now one checks that, except when P is 8#5 or 64#162, an automorphism of P of order 3 does *not* fix $\Omega_1(P)$, and similarly an automorphism of 64#153 of order 7 does not fix $\Omega_1(P)$.

Lemma 9.2. *The three groups 8#5 (Q_8), 64#162, and 64#187 (G_2) all satisfy Conjecture C.*

To see this, we look at each group in turn.

If $P = Q_8$ with center C, then $Out_0(P)$ is the symmetric group of order 6, and there is a 2 dimensional $\mathbb{F}_2[Out_0(P)]$ irreducible S. As $\mathbb{F}_2[Out_0(P)]$–modules, $S \simeq H^1(P/\Phi(P)) = H^1(P/C) = P_C H^1(P)$, and we are done.

If P is 64#162 with center C, the order of $Out_0(P)$ has the form $3 \cdot 2^a$. It follows that there must be exactly one irreducible $\mathbb{F}_2[Out_0(P)]$–module S on which an element of order 3 acts nontrivially. $H^1(P/\Phi(P)) = H^1(P/C) = P_C H^1(P) = H^1(P)$ is 4 dimensional, with basis $\{z, y, x, w\}$ as in [CTVZ]. Examining the action of the generating automorphisms on $H^1(P)$, one sees that the 2 dimensional subspace with basis $\{x + z, w\}$ is an $Out_0(P)$–submodule acted on nontrivially by an automorphism of order 3. Thus this submodule must be isomorphic to S, and we are done.

If P is 64#187 with center C, the order of $Out_0(P)$ has the form $5 \cdot 2^a$. It follows that there must be exactly one irreducible $\mathbb{F}_2[Out_0(P)]$–module S on which an element of order 5 acts nontrivially. $H^1(P/\Phi(P)) = H^1(P/C) = P_C H^1(P) = H^1(P)$ is four dimensional, and is an irreducible $\mathbb{F}_2[\mathbb{Z}/5]$–module as seen in §6.3. Thus $P_C H^1(P) \simeq S$, and we are done.

In summary, in this subsection we have checked

Proposition 9.3. *If P is an indecomposable 2–central subgroup of order dividing 64, with $C = \Omega_1(P)$, then $|Out_0(P)|$ has the form 2^a, $3 \cdot 2^a$, or $5 \cdot 2^a$, and all irreducible $\mathbb{F}_2[Out_0(P)]$–modules occur as submodules of $P_C H^0(P) = \mathbb{F}_2$ or $P_C H^1(P) = H^1(P/C)$.*

9.2. **On $Out_0(P \times Q)$.** The following Kunneth formula for primitives is easily verified.

Lemma 9.4. *If $C < P$ and $D < Q$ are central elementary abelian subgroups, then*

$$P_{C \times D} H^*(P \times Q) = P_C H^*(P) \otimes P_D H^*(Q).$$

Definition 9.5. Call a pair (P, Q) of 2–central groups *good* if the algebra homomorphism

$$\mathbb{F}_2[Out_0(P) \times Out_0(Q)] \to \mathbb{F}_2[Out_0(P \times Q)]$$

induces an isomorphism on their semisimple quotients.

In general, given two finite groups G and H, the irreducible $\mathbb{F}_2[G \times H]$ modules will be direct summands of modules of the form $S \otimes T$, where S is an irreducible $\mathbb{F}_2[G]$–module, and T is an irreducible $\mathbb{F}_2[H]$–module. Thus the lemma implies

Corollary 9.6. *If a pair (P, Q) of 2–central groups is good, and both P and Q satisfy Conjecture C, so does $P \times Q$.*

Now we aim to prove that pairs of 2–central groups are frequently good. The following is our key observation.

Lemma 9.7. *Let P and Q be 2–central. Given*

$$\alpha = \begin{pmatrix} \alpha_P & \alpha_{P,Q} \\ \alpha_{Q,P} & \alpha_Q \end{pmatrix} : P \times Q \to P \times Q$$

in $Aut_0(P \times Q)$, the following conditions on the component homomorphisms must hold.

(a) $\Omega_1(P) \subseteq Ker(\alpha_{Q,P})$, and $\Omega_1(Q) \subseteq Ker(\alpha_{P,Q})$.

(b) $Im(\alpha_{Q,P}) \subseteq Z(Q)$, and $Im(\alpha_{P,Q}) \subseteq Z(P)$.

(c) $\alpha_P \in Aut_0(P)$, and $\alpha_Q \in Aut_0(Q)$.

Proof. As α must be the identity when restricted to $\Omega_1(P \times Q) = \Omega_1(P) \times \Omega_1(Q)$, condition (a) is clear. Furthermore, α_P and α_Q must be the identity when respectively restricted to $\Omega_1(P)$ and $\Omega_1(Q)$.

As α is a homomorphism, it follows that $Im(\alpha_{Q,P})$ and $Im(\alpha_Q)$ must commute, and similarly $Im(\alpha_{P,Q})$ and $Im(\alpha_P)$ must commute.

It follows that both (b) and (c) will follow if we can show that α_P and α_Q are isomorphisms.

We are assuming that α is invertible. Thus there exists

$$\beta = \begin{pmatrix} \beta_P & \beta_{P,Q} \\ \beta_{Q,P} & \beta_Q \end{pmatrix} : P \times Q \to P \times Q$$

such that $\beta \circ \alpha : P \times Q \to P \times Q$ is the identity. In particular,

9.8. The composite $P \xrightarrow{\alpha_P, \alpha_{Q,P}} P \times Q \xrightarrow{\beta_P \cdot \beta_{P,Q}} P$ is the identity.

We will show that $\beta_P \circ \alpha_P : P \to P$ is epic. Let $\gamma = \beta_{P,Q} \circ \alpha_{Q,P} : P \to P$. Then 9.8 says

$$x = \beta_P(\alpha_P(x))\gamma(x) \text{ for all } x \in P.$$

By condition (a), $\Omega_1(P) \subseteq Ker(\gamma)$, and so the order of $\gamma(x)$ will be strictly less than the order of x for all $x \neq e \in P$. Thus we can prove that $\beta_P \circ \alpha_P$ is epic by induction on the order of x, as follows. Given $x \in P$, there exists z such that $\beta_P(\alpha_P(z)) = \gamma(x)$. But then

$$\beta_P(\alpha_P(xz)) = \beta_P(\alpha_P(x))\beta_P(\alpha_P(z)) = \beta_P(\alpha_P(x))\gamma(x) = x.$$

$\square$

Theorem 9.9. *A pair* (P, Q) *of 2–central groups is good if every homomorphism* $P/\Omega_1(P) \to Z(Q)$ *has image contained in* $\Phi(Q)$.

Proof. Let $\widetilde{Aut_0}(P)$ denote the image of $Aut_0(P) \to Aut(P/\Phi(P))$. Proposition 6.4 implies that $\mathbb{F}_2[Aut_0(P)] \to \mathbb{F}_2[\widetilde{Aut_0}(P)]$ induces an isomorphism on semisimple quotients, and similarly with P replaced by Q and $P \times Q$.

Using parts (a) and (b) of the lemma, the hypothesis that every homomorphism $P/\Omega_1(P) \to Z(Q)$ has image contained in $\Phi(Q)$ implies that for all $\alpha \in Aut_0(P \times Q)$, the component $\alpha_{Q,P} : P \to Q$ induces the zero map $P/\Phi(P) \to Q/\Phi(Q)$. Again using the lemma, it follows that

$$\widetilde{Aut_0}(P \times Q) = \begin{pmatrix} \widetilde{Aut_0}(P) & B \\ 0 & \widetilde{Aut_0}(Q) \end{pmatrix},$$

where B is a quotient of $\mathrm{Hom}(Q/\Omega_1(Q), Z(P))$, and is thus a 2–group.

The inclusion $\widetilde{Aut_0}(P) \times \widetilde{Aut_0}(Q) \to \widetilde{Aut_0}(P \times Q)$ thus has a retract with kernel a 2–group, and so induces an isomorphism on the semisimple quotients of the associated $\mathbb{F}_2$ group rings. $\square$

Corollary 9.10. *A pair* (P, Q) *of 2-central groups is good if any of the following conditions hold.*

(a) $Z(Q) \subseteq \Phi(Q)$.

(b) P *is elementary abelian.*

(c) $P/[P, P]\Omega_1(P)$ *is elementary abelian, and* Q *has no* $\mathbb{Z}/2$ *summands.*

(d) $Q = \mathbb{Z}/2^t$ *and* P *has exponent dividing* 2^t.

In summary, we have shown

Proposition 9.11. *If* P *and* Q *satisfy any of the conditions in the last corollary, and both satisfy Conjecture C, then* $P \times Q$ *also satisfies Conjecture C.*

9.3. **Many 2–central groups satisfy Conjecture C.** When one does a census of indecomposable noncyclic 2-central groups, one finds 1 of order 8, 2 of order 16, 9 of order 32, and 41 of order 64.[7] Condition (a) of Corollary 9.10 holds for all but one of these groups.

Proposition 9.12. *Let* Q *be an indecomposable 2–central group of order dividing 64. Then* $Z(Q) \subseteq \Phi(Q)$ *unless* Q *is cyclic or* $64\#30$. *The group* $64\#30$ *has exponent 4, and thus satisfies the condition for* P *listed in (c) of Corollary 9.10.*

Checking this is made easier by the observation that a 2-central group Q has no $\mathbb{Z}/2$ summands exactly when $\Omega_1(Q) \subseteq \Phi(Q)$. Thus $Z(Q) \subseteq \Phi(Q)$ whenever $Z(Q)$ is elementary abelian. This is the case for all but 11 of the 53 indecomposable noncyclic 2-central groups in question.

We can now easily prove Theorem 2.12, which we restate here.

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[7]Group $32\#25$ is mistakenly listed at indecomposable in both [Ca] and [CTVZ]. It is isomorphic to $\mathbb{Z}/2 \times Q_{16}$.

Theorem 9.13. *Conjecture C is true for all 2–central groups that can be written as the product of groups of order dividing 64.*

Proof. By Proposition 9.3, all indecomposable 2–central groups of order dividing 64 satisfy Conjecture C. Now suppose a 2–central group P is a product $P = A \times B \times C$ where A is abelian with no $\mathbb{Z}/2$ summands, B is a product of noncyclic indecomposable summands of order dividing 64, and C is elementary abelian. Repeatedly using condition (d) of Corollary 9.10, one deduces that A satisfies Conjecture C.[8] Repeatedly using conditions (a) and (c), one then deduces that $A \times B$ satisfies Conjecture C, and finally that $P = A \times B \times C$ does, thanks to condition (b).   □

10. A discussion of Example 2.14

In this section we discuss in detail Example 2.14, and use this to illustrate Theorem 2.8 and its proof. So let P be the group 64#108: all of our information about this group is available on [Ca].

10.1. **Subgroup structure.** The commutator subgroup $Z = [P, P]$ has order 2. The center C is elementary abelian of rank 2, and $C = \Phi(P)$, so $Z < C$ and P/C is elementary abelian of rank 4. There is a unique maximal elementary abelian group V of rank 3, and its centralizer $K = C_P(V)$ has order 32, so that $W_P(V) = G/K \simeq \mathbb{Z}/2$. More precisely, the 2–central group K is isomorphic to $(\mathbb{Z}/2)^2 \times Q_8$, with Q_8 embedded so that $V \cap Q_8 = Z$.[9]

We have the following picture of $\mathcal{A}^C(P)$:

$$(10.1) \qquad\qquad C \longrightarrow V \; \circlearrowright \; \mathbb{Z}/2$$

and from this it is already clear that $\mathcal{H}^*(P) = H^*(V)^{\mathbb{Z}/2}$.

10.2. $H_1(P)$ **as an** $\mathbb{F}_2[Out(P)]$**–module.** The group $Out(P)$ has order $768 = 3 \cdot 2^8$. From this, one can deduce that $\mathbb{F}_2[Out(P)]$ has precisely two irreducible modules, the trivial module '1', and another one which we will call 'S' of dimension at least two. Furthermore, S occurs as a composition factor in an $\mathbb{F}_2[Out(P)]$–module M if and only if any element of order 3 in $Out(P)$ acts nontrivially on M.

Now we consider $H_1(P) = P/C$ as an $\mathbb{F}_2[Out(P)]$–module. It is nontrivial by Proposition 6.4, thus S occurs as a composition factor. Notice that each of the subgroups C, V, and K are characteristic, and recall that $Inn(P)$ acts trivially on $H_1(P)$. Thus there is a composition series of $\mathbb{F}_2[Out(P)]$–modules:

$$V/C \subset K/C \subset P/C,$$

with composition factors V/C, K/V, and P/K. By dimension considerations, V/C and P/K are copies of the trivial module, and thus K/V must be S, which is thus two dimensional.

10.3. $H^*(V)$ **as an** $\mathbb{F}_2[Aut(P)]$**–module.** An automorphism of P order 3 fixes V.[10] Thus, as an $Aut(P)$–module, $H^*(V)$ has only trivial composition factors.

---

[8]Indeed $Aut_0(A)$ is a 2–group, as is well known.

[9]In terms of the generators $g_1, \ldots, g_6$ on the website, $Z = \langle g_6 \rangle, C = \langle g_3, g_6 \rangle, V = \langle g_2, g_3, g_6 \rangle,$ $Q_8 = \langle g_4, g_5, g_6 \rangle,$ and $K = \langle g_2, g_3, g_4, g_5, g_6 \rangle$. K is maximal subgroup #11.

[10]See automorphism #7 on the Carlson website.

10.4. $P_V H^*(K)$ and $P_C H^*(K)$. We have maps of unstable algebras equipped with with $Aut(P)$ action:

(10.2) $$P_V H^*(K) \hookrightarrow P_C H^*(K) \overset{j^*}{\longleftarrow} P_C H^*(P),$$

where $j : K \to P$ is the inclusion. In degree 1, this reads

$$\mathrm{Hom}(K/V, \mathbb{F}_2) \hookrightarrow \mathrm{Hom}(K/C, \mathbb{F}_2) \leftarrow \mathrm{Hom}(P/C, \mathbb{F}_2),$$

and so we see that j^* is onto in degree 1, and that $P_V H^1(K)$ is a copy of S.

The maps of pairs $(Q_8, Z) \to ((\mathbb{Z}/2)^2 \times Q_8, (\mathbb{Z}/2)^2 \times Z) = (K, V)$ induces an isomorphism of algebras:

$$P_V H^*(K) \simeq P_Z H^*(Q_8).$$

The algebra $P_Z H^*(Q_8)$ is familiar: the calculation of $H^*(Q_8)$ using the Serre spectral sequence associated to $Z \to Q_8 \to Q_8/Z$ reveals that $P_Z H^*(Q_8) = \mathrm{Im}\{H^*(Q_8/Z) \to H^*(Q_8)\} = B^*$, where B^* is the Poincaré duality algebra $\mathbb{F}_2[x, w]/(x^2 + xw + w^2, x^2 w + xw^2)$, where x and w both have degree 1. As an $\mathbb{F}_2[Aut(P)]$–module, $P_V H^*(K) = B^*$ is given by

$$B^d = \begin{cases} 1 & \text{if } d = 3 \\ S & \text{if } d = 2 \\ S & \text{if } d = 1 \\ 1 & \text{if } d = 0. \end{cases}$$

From this we learn that $P_C H^*(K) \simeq B^*[y]$ where y is also in degree 1, and thus is generated by elements in degree 1. It follows that $j^* : P_C H^*(P) \to P_C H^*(K)$ is onto, and then that $Inn(P)$ acts trivially on both $P_V H^*(K)$ and $P_C H^*(K)$.

10.5. $\bar{R}_d H^*(P)$. Applying the formula for $\bar{R}_d H^*(P)$ in Proposition 7.1 to our group, we see that, for all d, there is a pullback diagram of graded $\mathbb{F}_2[Out(P)]$–modules:

$$
\begin{array}{ccc}
\bar{R}_d H^*(P) & \longrightarrow & H^*(C) \otimes P_C H^d(P) \\
\downarrow & & \downarrow {\scriptstyle 1 \otimes j^*} \\
(H^*(V) \otimes P_V H^d(K))^{Inn(P)} & \longrightarrow & H^*(C) \otimes P_C H^d(K).
\end{array}
$$

By our comments above, this simplifies: for all d, there is a pullback diagram of unstable $\mathbb{F}_2[Out(P)]$–modules:

$$
\begin{array}{ccc}
\bar{R}_d H^*(P) & \longrightarrow & H^*(C) \otimes P_C H^*(P) \\
\downarrow & & \downarrow {\scriptstyle 1 \otimes j^*} \\
H^*(V)^{\mathbb{Z}/2} \otimes P_V H^*(K) & \longrightarrow & H^*(C) \otimes P_C H^*(K).
\end{array}
$$

As j^* is onto, we conclude

Proposition 10.1. *For all d, there is a short exact sequence of unstable $\mathbb{F}_2[Out(P)]$–modules $0 \to H^*(C) \otimes \ker j^d \to \bar{R}_d H^*(P) \to H^*(V)^{\mathbb{Z}/2} \otimes P_V H^d(K) \to 0$, where $H^*(C)$ and $H^*(V)^{\mathbb{Z}/2}$ have only trivial composition factors, while $P_V H^*(K) \simeq B^*$.*

We note that in the notation of §7, $\text{Ess}^*(C) = \ker j^*$ and $\text{Ess}^*(V) = P_V H^*(K)$.

From Proposition 10.1, one can already place the 3's occurring in the table in Example 2.14. The 2's will be determined by knowing the composition factors of the ideal $\ker j^* \subset P_C H^*(P)$. This we discuss next.

10.6. $P_C H^*(P)$ **and the ideal** $\ker j^*$. Obviously, $\ker j^0 = 0$, and we also know that $\ker j^1$ is one dimensional. Using more detailed information about $H^*(P)$, we outline how all of the composition factors of $\ker j^*$ can be determined.

$H^*(P)$ is generated by classes z, y, x, w, v, u, t of degrees 1,1,1,1,2,5,8 respectively, and the kernel of the restriction $H^*(P) \to H^*(K)$ is identified as the ideal (z).[11] Furthermore, this ideal is a free module on $\mathbb{F}_2[v, t]$ on an explicit basis of 24 monomials in the generators of degree 1.

All the degree 1 generators are $H^*(C)$–primitive, while v and t restrict to algebraically independent elements of $H^*(C)$. It follows that $\ker j^* = P_C H^*(P) \cap (z)$ is precisely the $\mathbb{F}_2$–span of this basis of monomials.

This allows us to conclude the following.

Firstly $P_C H^*(P) \subset H^*(P)$ is precisely the subalgebra generated by the classes in degree 1. Explicitly, $P_C H^*(P)$ is

$$A^* = \mathbb{F}_2[z, y, x, w]/(z^2, zy + x^2 + xw + w^2, zy^2 + x^2 w + xw^2, y^2 w^3 + yw^4 + w^5).$$

For our purposes, this can be simplified as follows. From the second and third relations, one can deduce that $x^3 = z(yx + y^2)$ and $w^3 = z(yw + y^2)$. Using these and the first relation, one deduces that $y^2 w^3 + yw^4 + w^5 = zy^4$. So we have

$$A^* = \mathbb{F}_2[z, y, x, w]/(z^2, zy^4, zy + x^2 + xw + w^2, zy^2 + x^2 w + xw^2).$$

Secondly, $\ker j^* = \Sigma(A^*/Ann(z))$, as unstable $\mathbb{F}_2[Out(P)]$–modules. Since $z^2 = zy^4 = 0$, $A^*/\text{Ann}(z)$ is a quotient of $A^*/(z, y^4) = \mathbb{F}_2[y, x, w]/(y^4, x^2 + xw + w^2, x^2 w + xw^2) = B^*[y]/(y^4)$. Inspection of the degrees of the monomial basis shows that $A^*/Ann(z)$ has dimension 1,3,5,6,5,3,1 in degrees 0,1,2,3,4,5,6. This agrees with $B^*[y]/(y^4)$, and we can conclude that $Ann(z) = (z, y^4)$, and then

Proposition 10.2. $\ker j^* \simeq \Sigma(B^*[y]/(y^4))$ *as unstable* $\mathbb{F}_2[Out(P)]$–*modules.*

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[11]See the description of the depth essential cohomology on the website.

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